Lifting Commutation Relations in Cuntz Algebras

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Lifting and Perturbation Problems

Lifting Problem: If we have elements in a quotient of a C*-algebra satisfying some relations, can they be lifted to elements satisfying the same relations?

Perturbing problem: Given some elements in a C*-algebra satisfying some relations, can they be moved in the C*-algebra to other elements satisfying these relations and some additional relations?

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Perturbing problem: Given some elements in a C*-algebra satisfying some relations, can they be moved in the C*-algebra to other elements satisfying these relations and some additional relations?

Partial Lifting Problem: Under above conditions, can we lift "part way"?

Weak Perturbing Problem: Can we do the perturbation if the given elements "almost" satisfy the additional relations?

Study these questions when the additional relations are commutation relations.

Kirchberg Algebras

Definition:

A *Kirchberg algebra* is a separable, unital, purely infinite simple nuclear C*-algebra in the UCT class.

"Purely infinite" means: every hereditary C*-subalgebra contains a projection equivalent to the identity.

Cuntz Algebras

The *Cuntz algebra* O_n is the universal (unital) C*-algebra generated by n isometries

$$s_1, \ldots, s_n$$

with mutually orthogonal range projections adding to the identity.

Relations:

$s_j^* s_j = 1$ for all j.
$s_k^* s_j = 0$ if $j \neq k$.
$\sum_{j=1}^n s_j s_j^* = 1$.

O_∞ is the universal (unital) C*-algebra generated by a sequence of isometries

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Theorem:

The Cuntz algebras are Kirchberg algebras.

Properties of Kirchberg Algebras

Let A be a Kirchberg algebra.

1. The Elliott invariant triple $L(A) = (K_0(A), K_1(A), [1_A])$ is a complete isomorphism invariant for A (among Kirchberg algebras). In fact, every homomorphism of Elliott invariants is realized by a unital *-homomorphism.
2. Every triple (G_0, G_1, u), where G_0, G_1 are countable abelian groups and $u \in G_0$, occurs as $L(A)$ for a (unique) Kirchberg algebra A.
3. Two nonzero projections in A are equivalent if and only if they have the same K_0-class.
4. Every K_0-class is represented by a nonzero projection in A.
5. The natural map from $U(A)/U_0(A)$ to $K_1(A)$ is an isomorphism.

6. Every unital C*-algebra Morita equivalent to A is a Kirchberg algebra. In particular, $M_n(A)$ for any n and pAp for any nonzero projection p in A are Kirchberg algebras.
7. Inductive limits and tensor products of Kirchberg algebras are Kirchberg algebras.

6. Every unital C*-algebra Morita equivalent to A is a Kirchberg algebra. In particular, $M_n(A)$ for any n and pAp for any nonzero projection p in A are Kirchberg algebras.
7. Inductive limits and tensor products of Kirchberg algebras are Kirchberg algebras.

$L(O_n) = (\mathbb{Z}_{n-1}, 0, 1)$ if n is finite.
$L(O_\infty) = (\mathbb{Z}, 0, 1)$.

In particular, the K-theory of O_2 is trivial.

Isomorphism of Tensor Products

1. $O_2 \otimes O_2 \cong O_2$.
2. $A \otimes O_2 \cong O_2$ for any Kirchberg algebra A.
3. $O_\infty \otimes O_\infty \cong O_\infty$.
4. $A \otimes O_\infty \cong A$ for any Kirchberg algebra A.
5. $O_2^n \cong O_2^\infty \cong O_2$ and $O_\infty^n \cong O_\infty^\infty \cong O_\infty$.

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5. $O_2^n \cong O_2^\infty \cong O_2$ and $O_\infty^n \cong O_\infty^\infty \cong O_\infty$.

However, it is doubtful that there are *explicit* isomorphisms!

Leavitt Path Algebras

The *Leavitt path algebra* L_n is the generic example of an algebra A for which the rank n free left A-module A^n is isomorphic to A.

More generally, Leavitt path algebras can be defined for directed graphs.

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Proposition:

The Leavitt path algebra L_n (over $\mathbb{C}$) is isomorphic to the dense *-subalgebra of O_n generated by $s_1, \ldots, s_n$.

Other Leavitt path algebras closely resemble Cuntz-Krieger algebras or more general Kirchberg algebras.

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$L_2 \odot L_2$ is *not* isomorphic to L_2.

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Thus there can be no isomorphism of $O_2 \otimes O_2$ and O_2 given by an algebraic formula.

This does not *quite* imply that there is no explicit isomorphism.

Absolute Retracts and Absolute Neighborhood Retracts

Definition:

A space X is an *absolute retract* (AR) if, whenever (Y, Z) is a space pair, and $\phi : Z \to X$ is a map, then ϕ extends to a map $\psi : Y \to X$.

Definition:

A space X is an *absolute neighborhood retract* (ANR) if, whenever (Y, Z) is a space pair, and $\phi : Z \to X$ is a map, then ϕ extends to a map $\psi : U \to X$ for some neighborhood U of Z in Y.

An ANR is locally contractible. A locally contractible finite-dimensional space is an ANR. An ANR is an AR if and only if it is contractible.

Alternate Characterization of ANR's

Proposition:

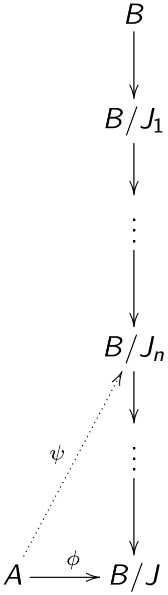
A space X is an ANR if and only if, whenever Y is a space and (Z_n) a decreasing sequence of closed subspaces with $Z = \cap_n Z_n$, and $\phi : Z \to X$ is a map, then ϕ extends to a map $\psi : Z_n \to X$ for some sufficiently large n.

Semiprojective C*-Algebras

When the alternate characterization of ANR's is rephrased by "turning the arrows around," we obtain the definition of a semiprojective C*-algebra:

Definition:

A separable C*-algebra A is *semiprojective* if, whenever B is a C*-algebra, (J_n) an increasing sequence of closed ideals of B with $J = [\cup_n J_n]^-$, and $\phi : A \to B/J$ is a *-homomorphism, there is a *-homomorphism $\psi : A \to B/J_n$ for some sufficiently large n, such that $\phi = \pi_J \circ \psi$. The map ψ is called a *partial lift* of ϕ.


B
B/J_1
B/J_n
ψ
A
ϕ
B/J


SEMIPROJECTIVITY
UP THE STAIRS

Examples

Easy examples:

- Finite-dimensional C*-algebras
- $C([0,1])$, $C(\mathbb{T})$
- The Toeplitz algebra, Cuntz (O_n, n finite) and Cuntz-Krieger algebras

Harder example: O_∞

Theorem:

Every semiprojective Kirchberg algebra has finitely generated K-theory.

Conjecture:

Every Kirchberg algebra with finitely generated K-theory is semiprojective.

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Theorem (D. Enders):

A Kirchberg algebra is semiprojective if and only if its K-theory is finitely generated.

Stable Relations

There is a theory of universal C*-algebras on a set of generators and relations. Relations are typically algebraic relations, or more generally of the form

$$\|p(x_{i_1}, \ldots, x_{i_n}, x_{i_1}^*, \ldots, x_{i_n}^*)\| \leq \eta$$

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In a semiprojective C*-algebra, relations are "stable": if A is semiprojective and is the universal C*-algebra on a finite set $\{x_1, \ldots, x_n\}$ of generators and a set $\mathcal{R}$ of relations, and $\{y_1, \ldots, y_n\}$ is a set of elements in a C*-algebra B which approximately satisfy the relations, then the y_j can be moved a small amount to elements $z_1, \ldots, z_n$ in B which exactly satisfy the relations.

So semiprojective C*-algebras are "flexible" in the sense that an approximate homomorphism into another C*-algebra can be "corrected" to an exact homomorphism.

Commutation Relations are Unstable!

Example: Voiculescu Matrices in $\mathbb{M}_n$

$$U_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, V_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & \omega & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \omega^{n-1} \end{bmatrix}$$

where $\omega = e^{2\pi/n} \in \mathbb{C}$.

$U_n^* V_n U_n = \omega V_n$, so $\|U_n V_n - V_n U_n\| = |\omega - 1| < \frac{2\pi}{n}$.

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where $\omega = e^{2\pi/n} \in \mathbb{C}$.

$U_n^* V_n U_n = \omega V_n$, so $\|U_n V_n - V_n U_n\| = |\omega - 1| < \frac{2\pi}{n}$.

(Voiculescu, Exel-Loring) U_n and V_n cannot be closely approximated by commuting unitaries in $\mathbb{M}_n$.

Conversion to a lifting problem:
Let $B = \prod_n \mathbb{M}_n$

J_n the sequences in B vanishing after the n'th coordinate

$J = [\cup_n J_n]^-$ the sequences in B vanishing at infinity.

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Then u and v give commuting unitaries in B/J which cannot be lifted to commuting unitaries in B/J_n for any n.

Conclusion:

$C(\mathbb{T}^2)$ is not semiprojective.

The definition of semiprojectivity can be made in any category of (separable) C*-algebras. Obviously, $C(X)$ is semiprojective in the category of unital commutative C*-algebras if and only if X is an ANR. But if X is an ANR, $C(X)$ is not necessarily semiprojective in the category of all separable C*-algebras.

Example: Commutation relations are not generally partially liftable. In fact, $C([0,1]^2)$ is not semiprojective, even though $[0,1]^2$ is an ANR (in fact, an AR).

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Theorem (Sørensen-Thiel):

$C(X)$ is semiprojective in the category of general separable C*-algebras if and only if X is an ANR with $dim(X) \leq 1$.

Products of C*-Algebras

Let A and B be (separable) unital C*-algebras. Consider:

1. The *free product* $A * B$, the universal C*-algebra generated by copies of A and B with no relations. (This C*-algebra is nonunital.)
2. The *unital free product* $A *_{\mathbb{C}} B$, the universal C*-algebra generated by copies of A and B with a common unit and no other relations.
3. The *soft tensor product* $A \circledast_{\epsilon} B$, the universal C*-algebra generated by copies of A and B with a common unit such that the generators of A approximately commute within ϵ with the generators of B.
4. The *tensor product* $A \otimes B$, the universal C*-algebra generated by copies of A and B which commute and have a common unit.

There are obvious quotient maps

$$A * B \to A *_{\mathbb{C}} B \to A \circledast_{\epsilon} B \to A \circledast_{\epsilon'} B \to A \otimes B$$

for $0 < \epsilon' < \epsilon$.

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Main Question:

When are there splittings (cross sections) for these quotient maps?

The short answer is "rarely."

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Main Question:

When are there splittings (cross sections) for these quotient maps?

The short answer is "rarely."

The question can be rephrased as a perturbation question: If A and B are [unital] subalgebras of a C*-algebra D [which approximately commute], can they be perturbed in a "functorial" way to other copies of A and B which exactly commute [or commute better]?

Definition:

Let A and B be unital C*-algebras.

(A, B) has *full splitting* if the quotient map

$$A * B \to A \otimes B$$

has a splitting.

(A, B) has *full unital splitting* if the quotient map

$$A *_{\mathbb{C}} B \to A \otimes B$$

has a splitting.

(A, B) has *partial splitting* if the quotient map

$$A \circledast_\epsilon B \to A \otimes B$$

has a splitting for some $\epsilon > 0$ (hence also for all smaller ϵ).

Thus the Main Question for a pair (A, B) splits into three questions:

1. Does (A, B) have a full [unital] splitting?
2. Does (A, B) have a partial splitting?
3. If (A, B) has a partial splitting, does it have a full [unital] splitting?

Obstructions to 2 are related to semiprojectivity.

There are severe K-theoretic obstructions to 3, and hence 1. But not all obstructions to 3 are K-theoretic.

Proposition:

If $A \otimes B$ is semiprojective, then (A, B) has partial splitting. If A and B are semiprojective, then (A, B) has partial splitting if and only if $A \otimes B$ is semiprojective.

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What about the quotient map from $O_n \circledast_\epsilon O_n$ to $O_n \otimes O_n$?

Semiprojectivity is not enough to give a full (unital) splitting:

Consider the simplest nontrivial C*-algebra $\mathbb{C}^2$. Then $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ is semiprojective. $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ is the universal unital C*-algebra generated by two projections, is isomorphic to the continuous functions from $[0,1]$ to $\mathbb{M}_2$ which are diagonal at the endpoints.

From this description it is easy to see that there is no splitting for the quotient map from $\mathbb{C}^2 *_{\mathbb{C}} \mathbb{C}^2$ to $\mathbb{C}^2 \otimes \mathbb{C}^2$ (which is just evaluation at the endpoints of $[0,1]$.) The obstruction is K-theoretic: there is no splitting at the K_0 level.

Projectivity of $A \otimes B$ (in the unital category) suffices, but is not necessary.

There is no partial splitting for $(C(\mathbb{T}), C(\mathbb{T}))$ (the Voiculescu matrices show this is impossible.)

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There is no partial splitting for $(C([0,1]), C([0,1]))$ by a similar argument, even though there is a splitting on the K-theory level.

One can ask whether the pair (A, B) *ever* has a full [unital] splitting (if neither A nor B is $\mathbb{C}$). Perhaps surprisingly, the answer is yes: it splits if A and B are certain Kirchberg algebras (but far from all pairs of Kirchberg algebras).

Notation: P is a separable unital C*-algebra with the following properties:

(i) P is semiprojective.

(ii) P is isomorphic to $P^n = \bigotimes_{k=1}^{n} P$ for any n and to $P^\infty = \bigotimes_{k=1}^{\infty} P$.

For example, $P = O_2$ or $P = O_\infty$. Are O_2 and O_∞ the only C*-algebras besides $\mathbb{C}$ satisfying the two conditions? They are the only Kirchberg algebras.

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Let A be a unital semiprojective C*-algebra which is P-absorbing, e.g. $A = O_2$ if $P = O_2$ and A is any semiprojective Kirchberg algebra if $P = O_\infty$. $A = P$ works in general.

Theorem:

Let Q be the full free product of A and a sequence of copies of P, i.e. Q is the universal C*-algebra generated by a copy of A and a sequence of copies of P with no relations. The canonical quotient map $\pi : Q \to A \otimes P^\infty$ splits.

$$\begin{array}{c} A * P * P * P * \cdots \\ \downarrow \\ A \otimes P \otimes P \otimes P \otimes \cdots \end{array}$$

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$$\downarrow$$
$$A \otimes P \otimes P \otimes P \otimes \cdots$$

Corollary:

For any $n \geq 1$, the quotient map from $A * P * \cdots * P$ (n copies of P) to $A \otimes P^n \cong A \otimes P$ splits. In particular, (A, P) has a full splitting.

Proof: For each n, let

$$Q_n = (A \otimes P^n) * P * P * \cdots$$

which is the universal C*-algebra generated by a copy of A and a sequence of copies of P such that the copy of A and the first n copies of P commute and have a common unit.

There is an obvious canonical quotient map $\pi_n : Q \to Q_n$ for all n. Let J_n be the kernel. Let $J = [\cup_n J_n]^-$. Then

$$Q/J \cong A \otimes P^\infty \cong A \otimes P \cong A$$

so Q/J is semiprojective, and the identity map on Q/J partially lifts to a homomorphism from Q/J to Q/J_n for some n.

$$(A \otimes P^n) * P * P * P * \cdots$$

$$\downarrow$$

$$(A \otimes P^n) \otimes P \otimes P \otimes P \otimes \cdots$$

Identify $A \otimes P^n$ with A.

In particular, we have:

Corollary:

(i) There is a splitting for the quotient map

$$O_2 * O_2 \to O_2 \otimes O_2 \ .$$

(ii) If A is any Kirchberg algebra with finitely generated K-theory, then there is a splitting for the quotient map

$$A * O_\infty \to A \otimes O_\infty \ .$$

Thus two copies of either O_2 or O_∞ in a C*-algebra D can be moved functorially in D to copies which commute and have a common unit.

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Thus two copies of either O_2 or O_∞ in a C*-algebra D can be moved functorially in D to copies which commute and have a common unit.

There is, however, probably not an *explicit* splitting.

K-Theoretic Obstructions

We restrict to A and B in the UCT class with finitely generated K-theory.

$$K_0(A *_{\mathbb{C}} B) \cong [K_0(A) \oplus K_0(B)]/\langle([1_A], -[1_B])\rangle$$
$$K_1(A *_{\mathbb{C}} B) \cong K_1(A) \oplus K_1(B)$$

$$K_0(A \otimes B) \cong [K_0(A) \otimes_{\mathbb{Z}} K_0(B)] \oplus [K_1(A) \otimes_{\mathbb{Z}} K_1(B)] \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K_1(B)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(A), K_0(B))$$

$$K_1(A \otimes B) \cong [K_0(A) \otimes_{\mathbb{Z}} K_1(B)] \oplus [K_1(A) \otimes_{\mathbb{Z}} K_0(B)] \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_0(A), K_0(B)) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_1(A), K_1(B))$$

Theorem:

(A, B) can have a full unital splitting only in the following situations:

(i) $L(A) = (0,0,0)$ or $(\mathbb{Z}, 0, 1)$, $L(B)$ arbitrary (or vice versa). In the first case $L(A \otimes B) = (0,0,0)$, and in the second $L(A \otimes B) \cong L(B)$.

(ii) $L(A) = (G_0, G_1, r)$, $L(B) = (H_0, H_1, s)$. In this case $L(A \otimes B) = (0,0,0)$.

(iii) $L(A) = (\mathbb{Z} \oplus G_0, G_1, (u, r))$, $L(B) = (\mathbb{Z} \oplus H_0, H_1, (w, s))$, where u is relatively prime to w and to the orders of H_0 and H_1, and w is relatively prime to the orders of G_0 and G_1.

(iv) $L(A) = (\mathbb{Z}, 0, u)$, $L(B) = (\mathbb{Z}^b \oplus H_0, H_1, (w, 0, \ldots, 0, s))$, where $b > 1$, and u is relatively prime to w and to the orders of H_0 and H_1.

In (ii)–(iv), G_0, G_1, H_0, H_1 are any finite abelian groups with the orders of G_0 and G_1 relatively prime to the orders of H_0 and H_1.

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Unknown possibilities:

3. (A, B), A and B semiprojective, $A \otimes B \cong O_2$ (e.g. (O_2, B), B semiprojective, in particular (O_2, O_n), $n > 2$; or (O_3, O_4).)
4. $(M_p(O_\infty), M_q(O_\infty))$, p and q relatively prime (and slight generalizations).
5. (A, B), A or B not semiprojective.

Before looking at $(M_2(O_\infty), M_3(O_\infty))$, consider the simpler case $(\mathbb{M}_2, \mathbb{M}_3)$. Is there a cross section for the quotient map

$$\mathbb{M}_2 *_{\mathbb{C}} \mathbb{M}_3 \to \mathbb{M}_2 \otimes \mathbb{M}_3 \cong \mathbb{M}_6 \text{ ?}$$

There is no K-theoretic obstruction: the Elliott invariant of both algebras is $(\mathbb{Z}, 0, 6)$.

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There is no K-theoretic obstruction: the Elliott invariant of both algebras is $(\mathbb{Z}, 0, 6)$.

But there is an *ordered* K-theory obstruction: the positive cone of $K_0(\mathbb{M}_2 *_{\mathbb{C}} \mathbb{M}_3) \cong \mathbb{Z}$ consists of 0 and the subsemigroup generated by 2 and 3; the scale consists of $\{0, 2, 3, 4, 6\}$. In particular, there is no unital C*-subalgebra of $\mathbb{M}_2 *_{\mathbb{C}} \mathbb{M}_3$ isomorphic to $\mathbb{M}_6$.

Rather surprisingly, the situation is much better for $(M_2(O_\infty), M_3(O_\infty))$:

Theorem:

There is a unital copy of $\mathbb{M}_6$ in $M_2(O_\infty) *_{\mathbb{C}} M_3(O_\infty)$; hence the ordering on $K_0(M_2(O_\infty) *_{\mathbb{C}} M_3(O_\infty))$ is the ordinary ordering.

So there is no ordered K-theoretic obstruction to a splitting.

Rather surprisingly, the situation is much better for $(M_2(O_\infty), M_3(O_\infty))$:

Theorem:

There is a unital copy of $\mathbb{M}_6$ in $M_2(O_\infty) *_{\mathbb{C}} M_3(O_\infty)$; hence the ordering on $K_0(M_2(O_\infty) *_{\mathbb{C}} M_3(O_\infty))$ is the ordinary ordering.

So there is no ordered K-theoretic obstruction to a splitting.

Does $M_2(O_\infty) *_{\mathbb{C}} M_3(O_\infty)$ contain a unital copy of $M_6(O_\infty)$?

Philosophical Conclusion: O_2 and O_∞ are "almost projective."