Homotopies and Liftings

Bruce Blackadar

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Conventions

"Space" $=$ "compact metrizable space."

"Map" $=$ "continuous function."

"Subspace" $=$ (usually) "closed subspace."

"Space pair" will mean a pair (Y, Z), where Y is a space and Z a (closed) subspace.

"Ideal" $=$ "two-sided ideal."

Definition:

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(ii) $[0, 1]$ is an AR (*Tietze Extension Theorem*).
(iii) $[0, 1]^n$ is an AR for any n, even $n = \aleph_0$ (Hilbert cube).

Definition:

A space X is an *absolute neighborhood retract* (ANR) if, whenever (Y, Z) is a space pair, and $\phi : Z \to X$ is a map, then ϕ extends to a map $\psi : U \to X$ for some neighborhood U of Z in Y.

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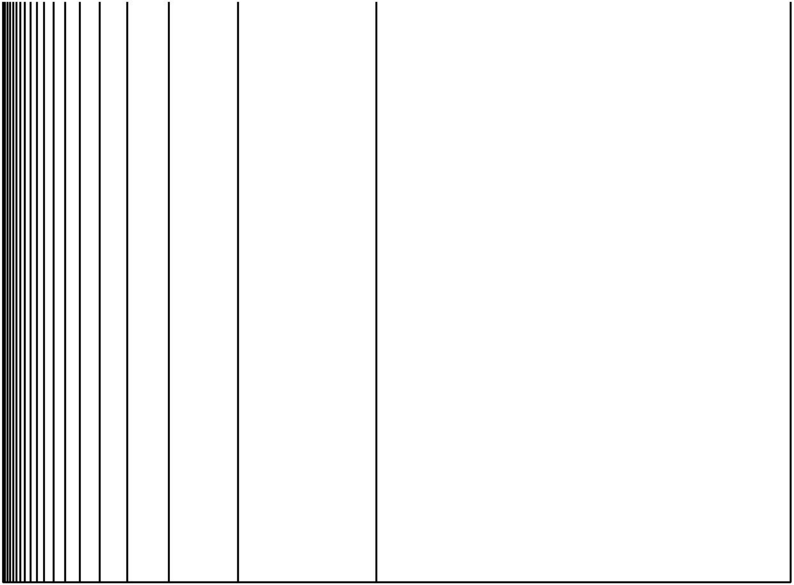
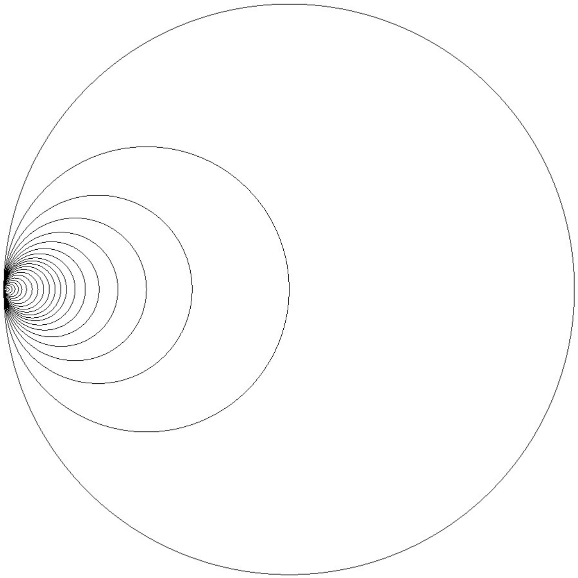
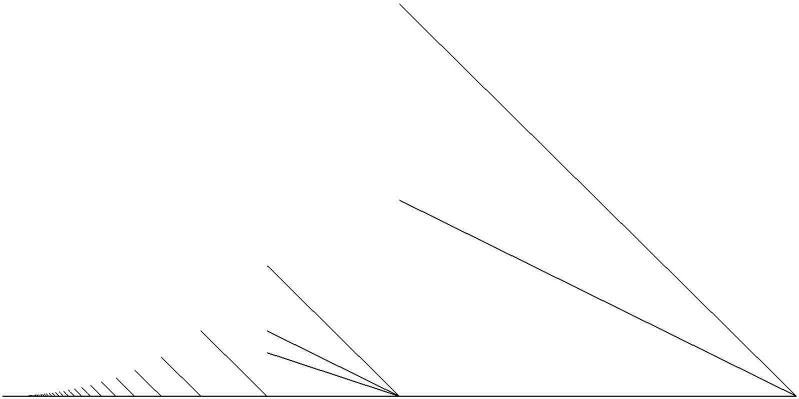
(v) Any polyhedron is an ANR. Every ANR is homotopy equivalent to a polyhedron.
(vi) Let $N = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq [0, 1]$. Then N is not an ANR since no neighborhood in $[0, 1]$ retracts.







In fact, every ANR is locally contractible. An ANR is an AR if and only if it is contractible.

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Alternate Characterization of ANR's

Proposition:

A space X is an ANR if and only if, whenever Y is a space and (Z_n) a decreasing sequence of closed subspaces with $Z = \cap_n Z_n$, and $\phi : Z \to X$ is a map, then ϕ extends to a map $\psi : Z_n \to X$ for some sufficiently large n.

Homotopies

If X and Y are spaces, and ϕ_0, ϕ_1 are maps from Y to X, then a *homotopy* from ϕ_0 to ϕ_1 is a “continuous deformation” of ϕ_0 to ϕ_1. Two equivalent ways to picture a homotopy:

(i) A path (ϕ_t) $(0 \leq t \leq 1)$ of maps from Y to X which is continuous in t for the topology of uniform convergence.

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(i) A path (ϕ_t) $(0 \leq t \leq 1)$ of maps from Y to X which is continuous in t for the topology of uniform convergence.
(ii) A continuous function $h : [0, 1] \times Y \to X$ such that $h(0, y) = \phi_0(y)$ and $h(1, y) = \phi_1(y)$ for all $y \in Y$.
For the correspondence, let $h(t, y) = \phi_t(y)$.

The Borsuk Homotopy Extension Theorem

Theorem:

Let X be an ANR, (Y, Z) a space pair, (ϕ_t) $(0 \leq t \leq 1)$ a uniformly continuous path of maps from Z to X (i.e. $h(t, z) = \phi_t(z)$ is a homotopy from ϕ_0 to ϕ_1). Suppose ϕ_0 extends to a map $\bar{\phi}_0$ from Y to X. Then there is a uniformly continuous path $\bar{\phi}_t$ of extensions of the ϕ_t to functions from Y to X (i.e. $\bar{h}(t, y) = \bar{\phi}_t(y)$ is a homotopy from $\bar{\phi}_0$ to $\bar{\phi}_1$). In particular, ϕ_1 extends to Y.

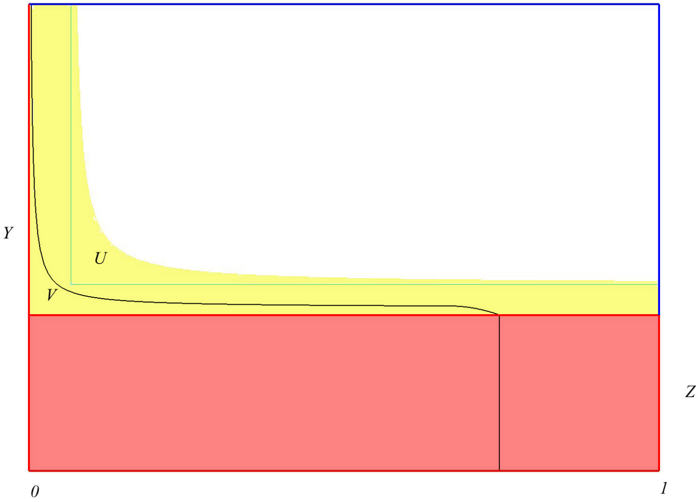
"Universal cofibration property" for maps into X.

Idea of Proof

Let $\tilde{Y} = [0,1] \times Y$, $\tilde{Z} = (\{0\} \times Y) \cup ([0,1] \times Z)$.

Dowker's Lemma:

Inside any neighborhood U of $\tilde{Z}$ in $\tilde{Y}$ is a subspace containing $\tilde{Z}$ which is homeomorphic to $\tilde{Y}$ under a homeomorphism which is the identity on $\tilde{Z}$.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C*-Algebras

C*-Algebras were originally algebras of operators on a Hilbert space:

Definition:

A (concrete) C*-algebra is a norm-closed *-algebra of bounded operators on a Hilbert space.

A concrete C*-algebra comes equipped with:
A complex vector space structure
A ring (algebra) structure
An adjoint operation (involution)
A complete norm.

Abstract C*-Algebras

C*-algebras can be abstractly characterized by a simple set of axioms:

Definition:

A C*-algebra is a complex Banach *-algebra A satisfying the *C*-axiom:*

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

"Complex Banach *-algebra" means: A has

A complex vector space structure

A ring (algebra) structure

An involution or *-operation: a conjugate-linear antiautomorphism of order 2

A complete norm which is submultiplicative: $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

Theorem (Gelfand-Naimark):

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It is another theorem of Gelfand and Naimark which eventually spawned the point of view of C*-algebra theory as "noncommutative topology":

"Theorem": The theory of commutative C*-algebras *is* topology.

Or, topology (of locally compact Hausdorff spaces) *is* the theory of commutative C*-algebras.

To be precise, let X be a space. Associate to X the algebra $C(X)$ of complex-valued continuous functions on X (the algebra of real-valued continuous functions works just as well).

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$C(X)$ has

An algebra structure with ordinary pointwise addition and multiplication of functions

An involution: complex conjugation

A complete norm which is submultiplicative and satisfies the C*-axiom:

$$\|f\| = \max_{t \in X} |f(t)| \ .$$

Thus:

Proposition:

$C(X)$ is a commutative C*-algebra.

If X and Y are spaces, and $\phi : X \to Y$ is a map, then ϕ defines a homomorphism ϕ_* from $C(Y)$ to $C(X)$ by $\phi_*(f) = f \circ \phi$.

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Two remarkable facts are true:

(i) X can be recovered from the algebraic structure of $C(X)$ (e.g. as the space of maximal ideals with the hull-kernel topology).

(ii) Every homomorphism from $C(Y)$ to $C(X)$ is of the form ϕ_* for some map $\phi : X \to Y$.

Theorem (Gelfand-Naimark):

Every (separable unital) commutative C*-algebra is $C(X)$ for a unique (compact metrizable) space X.

There are versions for nonseparable unital C*-algebras (compact Hausdorff spaces) and nonunital C*-algebras (locally compact Hausdorff spaces).

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The upshot is:

Theorem:

The correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of (compact metrizable) spaces and maps and the category of (separable unital) commutative C*-algebras and homomorphisms.

Turning Arrows Around

So any statement about spaces can in principle be converted into an equivalent statement about commutative C*-algebras by "turning arrows around."

Such statements often make sense, and are even sometimes true, for noncommutative C*-algebras too.

Notable example: Complex K-theory.

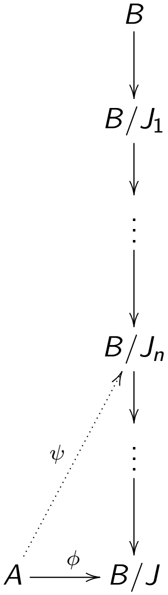
Thus C*-algebra theory is sometimes called "noncommutative topology."

Semiprojective C*-Algebras

When the alternate characterization of ANR's is rephrased by "turning the arrows around," we obtain the definition of a semiprojective C*-algebra:

Definition:

A separable C*-algebra A is *semiprojective* if, whenever B is a C*-algebra, (J_n) an increasing sequence of closed ideals of B with $J = [\cup_n J_n]^-$, and $\phi : A \to B/J$ is a *-homomorphism, there is a *-homomorphism $\psi : A \to B/J_n$ for some sufficiently large n, such that $\phi = \pi_J \circ \psi$. The map ψ is called a *partial lift* of ϕ.


B
B/J_1
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ψ
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SEMIPROJECTIVITY
UP THE STAIRS

Examples:

Finite-dimensional C*-algebras

$C([0,1])$, $C(\mathbb{T})$

The Toeplitz algebra, Cuntz and Cuntz-Krieger algebras

The dimension-drop algebras $\mathbb{I}_{p,q}$

$+$ many more!

The definition of semiprojectivity can be made in any category of (separable) C*-algebras. Obviously, $C(X)$ is semiprojective in the category of unital commutative C*-algebras if and only if X is an ANR. But if X is an ANR, $C(X)$ is not necessarily semiprojective in the category of all separable C*-algebras.

Example: Commutation relations are not generally partially liftable. In fact, $C([0,1]^2)$ is not semiprojective, even though $[0,1]^2$ is an ANR (in fact, an AR).

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Theorem (Sørensen-Thiel):

$C(X)$ is semiprojective in the category of general separable C*-algebras if and only if X is an ANR with $dim(X) \leq 1$.

Stable Relations

There is a theory of universal C*-algebras on a set of generators and relations. Relations are typically algebraic relations, or more generally of the form

$$\|p(x_{i_1}, \ldots, x_{i_n}, x_{i_1}^*, \ldots, x_{i_n}^*)\| \leq \eta$$

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In a semiprojective C*-algebra, relations are "stable": if A is semiprojective and is the universal C*-algebra on a finite set $\{x_1, \ldots, x_n\}$ of generators and a set $\mathcal{R}$ of relations, and $\{y_1, \ldots, y_n\}$ is a set of elements in a C*-algebra B which approximately satisfy the relations, then the y_j can be moved a small amount to elements $z_1, \ldots, z_n$ in B which exactly satisfy the relations.
So semiprojective C*-algebras are "flexible" in the sense that an approximate homomorphism into another C*-algebra can be "corrected" to an exact homomorphism.

The Homotopy Lifting Theorem

Corresponding statement for semiprojective C*-algebras:

Theorem:

Let A be a semiprojective C*-algebra, B a C*-algebra, I a closed ideal of B, (ϕ_t) $(0 \leq t \leq 1)$ a point-norm continuous path of *-homomorphisms from A to B/I. Suppose ϕ_0 lifts to a *-homomorphism $\bar{\phi}_0 : A \to B$, i.e. $\pi_I \circ \bar{\phi}_0 = \phi_0$. Then there is a point-norm continuous path $(\bar{\phi}_t)$ $(0 \leq t \leq 1)$ of *-homomorphisms from A to B beginning at $\bar{\phi}_0$ such that $\bar{\phi}_t$ is a lifting of ϕ_t for each t, i.e. the entire homotopy lifts. In particular, ϕ_1 lifts to a *-homomorphism from A to B.

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The commutative proof has no hope of working in the noncommutative case. We need a different approach.


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Pučko otvoreno učilište
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The General Chinese Remainder Theorem

Proposition:

Let R be a ring, and I and J ideals in R. Then the map

$$\phi : a \mapsto (a \bmod I, a \bmod J)$$

gives an isomorphism from $R/(I \cap J)$ onto the fibered product

$$P = \{(x, y) : x \in R/I, y \in R/J, \pi_1(x) = \pi_2(y)\} \subseteq (R/I) \oplus (R/J)$$

where $\pi_1 : R/I \to R/(I + J)$ and $\pi_2 : R/J \to R/(I + J)$ are the quotient maps.

P is the pullback of (π_1, π_2).

The usual Chinese Remainder Theorem is the special case where $I + J = R$, where P is just the direct sum $R/I \oplus R/J$.

In particular, to define a homomorphism from another ring into $R/(I \cap J)$, it suffices to give a compatible pair of homomorphisms into R/I and R/J.

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In the case $R = C(X)$, and (closed) ideals I and J correspond to closed sets Y and Z, there is a one-one correspondence between maps from $Y \cup Z$ to $\mathbb{C}$ and pairs (g, h), where $g : Y \to \mathbb{C}$ and $h : Z \to \mathbb{C}$ are maps which agree on $Y \cap Z$.

Partial Liftings with Specified Quotient

If B is a C*-algebra, (J_n) an increasing sequence of closed ideals of B, and $J = [\cup J_n]^-$, and A is a semiprojective C*-algebra, then any homomorphism $\phi : A \to B/J$ can be partially lifted to a homomorphism $\psi : A \to B/J_n$ for some sufficiently large n.

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But suppose we also have another closed ideal I of B and a homomorphism $\tilde{\phi}$ from A to B/I such that ϕ and $\tilde{\phi}$ agree mod $I + J$. Can we partially lift ϕ to ψ so that ψ agrees with $\tilde{\phi}$ mod $I + J_n$?

Theorem:

Let A be a semiprojective C*-algebra, B a C*-algebra, (J_n) an increasing sequence of closed ideals of B with $J = [\cup J_n]^-$, I another closed ideal of B, and $\phi : A \to B/J$ and $\tilde{\phi} : A \to B/I$ *-homomorphisms with $\pi_{I+J} \circ \phi = \pi_{I+J} \circ \tilde{\phi}$. Then for some sufficiently large n there is a *-homomorphism $\psi : A \to B/J_n$ such that $\pi_J \circ \psi = \phi$ and $\pi_{I+J_n} \circ \psi = \pi_{I+J_n} \circ \tilde{\phi}$.

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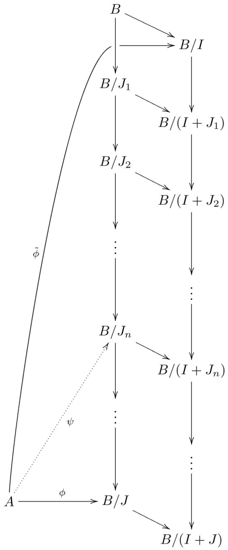
If X is an ANR, Y a space, Z_n a decreasing sequence of closed subspaces with $Z = \cap_n Z_n$, W another closed subspace, and $\phi : Z \to X$ and $\tilde{\phi} : W \to X$ are maps which agree on $Z \cap W$, then ϕ can be extended to $\psi : Z_n \to X$ for some n so that ψ agrees with $\tilde{\phi}$ on $Z_n \cap W$.
[Note that $Z \cup W = \cap_n (Z_n \cup W)$.]


B
B/I
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B/J_2
B/(I + J_2)
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B/J_n
B/(I + J_n)
ψ
A
φ
B/J
B/(I + J)

By the Chinese Remainder Theorem ϕ and $\tilde{\phi}$ define a homomorphism $\bar{\phi}$ from A to $B/(I \cap J)$, which partially lifts to a homomorphism $\bar{\psi}$ from A to $B/(I \cap J_n)$ for some n by semiprojectivity. The map $\bar{\psi}$ defines compatible homomorphisms $\psi : A \to B/J_n$ and $\tilde{\psi} : A \to B/I$. But we have $\tilde{\psi} = \tilde{\phi}$ since the reduction from $B/(I \cap J_n)$ to B/I factors through $B/(I \cap J)$, where $\bar{\psi}$ becomes $\bar{\phi}$.

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Technicality: It is not obvious that $\cup_n (I \cap J_n)$ is dense in $I \cap J$, but this follows from the uniqueness of norm on a C*-algebra.

Lifting Close Homomorphisms

If A and B are C*-algebras and I is a closed ideal of B, then a homomorphism from A to B/I need not lift in general to a homomorphism from A to B, even if A is semiprojective. But suppose $\phi : A \to B/I$ does lift to $\bar{\phi} : A \to B$, and ψ is another homomorphism from A to B/I which is close to ϕ in the point-norm topology. If A is semiprojective, does ψ also lift to B, and can the lift be chosen close to $\bar{\phi}$ in the point-norm topology? The answer is yes in the commutative category, but the commutative proof does not generalize to the noncommutative case. However, we can by a different argument obtain the same result for general semiprojective C*-algebras.

The next theorem is really the main result:

Theorem:

Let A be a semiprojective C*-algebra generated by a finite or countable set $\mathcal{G} = \{x_1, x_2, \ldots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathcal{G}$ is infinite. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that, whenever B is a C*-algebra, I a closed ideal of B, ϕ and ψ *-homomorphisms from A to B/I with $\|\phi(x_j) - \psi(x_j)\| < \delta$ for all j and such that ϕ lifts to a *-homomorphism $\bar{\phi} : A \to B$ (i.e. $\pi_I \circ \bar{\phi} = \phi$), then ψ also lifts to a *-homomorphism $\bar{\psi} : A \to B$ with $\|\bar{\psi}(x_j) - \bar{\phi}(x_j)\| < \epsilon$ for all j. (The δ depends on ϵ, A, and the set $\mathcal{G}$ of generators, but not on the B, I, ϕ, ψ.)

Outline of Proof

If the statement is false, then there is an $\epsilon > 0$ such that there are B_n, I_n, ϕ_n, ψ_n with

$$\|\phi_n(x_j) - \psi_n(x_j)\| < \frac{1}{n}$$

for all n and j, such that ϕ_n lifts but ψ_n does not. Let $B = \prod_n B_n$, $I = \prod_n I_n$, J_n the ideal of elements of B vanishing after the n'th term, $J = [\cup J_n]^- = \oplus_n B_n$. Let $\bar{\phi} : A \to B$ be defined by

$$\bar{\phi}(x) = (\bar{\phi}_1(x), \bar{\phi}_2(x), \dots)$$

and let $\phi = \pi_J \circ \bar{\phi} : A \to B/J$. There is also a homomorphism $\tilde{\phi}$ from A to $B/I \cong \prod_n (B_n/I_n)$ defined by

$$\tilde{\phi}(x) = (\psi_1(x), \psi_2(x), \dots) .$$

Then ϕ cannot be lifted to B/J_n compatibly with $\tilde{\phi}$, contradiction.

The Homotopy Lifting Theorem Again

Theorem:

Let A be a semiprojective C*-algebra, B a C*-algebra, I a closed ideal of B, (ϕ_t) $(0 \leq t \leq 1)$ a point-norm continuous path of *-homomorphisms from A to B/I. Suppose ϕ_0 lifts to a *-homomorphism $\bar{\phi}_0 : A \to B$, i.e. $\pi_I \circ \bar{\phi}_0 = \phi_0$. Then there is a point-norm continuous path $(\bar{\phi}_t)$ $(0 \leq t \leq 1)$ of *-homomorphisms from A to B beginning at $\bar{\phi}_0$ such that $\bar{\phi}_t$ is a lifting of ϕ_t for each t, i.e. the entire homotopy lifts. In particular, ϕ_1 lifts to a *-homomorphism from A to B.

Proof of the Homotopy Lifting Theorem

Let $\mathcal{G} = \{x_1, x_2, \dots\}$ be a countable set of generators for A, with $\|x_j\| \to 0$. Fix $\epsilon > 0$, say $\epsilon = 1$, and fix $\delta > 0$ satisfying the conclusion of the previous theorem for ϵ, A, $\mathcal{G}$.

Choose a finite partition $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ such that $\|\phi_s(x_j) - \phi_t(x_j)\| < \delta$ for all j whenever $s, t \in [t_{i-1}, t_i]$ for any i. There is such a partition since one only needs to consider finitely many x_j, the condition being automatic for any x_j with $\|x_j\| < \frac{\delta}{2}$.

Begin with $[0, t_1]$. Let $\tilde{B} = C([0, t_1], B)$, and J the ideal of $\tilde{B}$ consisting of functions $f : [0, t_1] \to I$ with $f(0) = 0$. Then

$$\tilde{B}/J \cong \{(f, b) \in C([0, t_1], B/I) \oplus B : f(0) = \pi_I(b)\} \ .$$

Define homomorphisms $\phi, \psi : A \to \tilde{B}/J$ by setting $\phi(x) = (f_x, \bar{\phi}_0(x))$, where $f_x(t) = \phi_0(x)$ for all t, and $\psi(x) = (g_x, \bar{\phi}_0(x))$, where $g_x(t) = \phi_t(x)$ for all t. We then have

$$\|\phi(x_j) - \psi(x_j)\| < \delta$$

for all j. Since ϕ lifts to a *-homomorphism from A to $\tilde{B}$ (e.g. by the constant function $\bar{\phi}_0$), ψ also lifts, defining a continuous path of lifts $(\bar{\phi}_t)$ of the ϕ_t for $0 \leq t \leq t_1$.

Now repeat the process on $[t_1, t_2]$, using the lift $\bar{\phi}_{t_1}$ as the starting point, and continue through all the intervals. After a finite number of steps the entire homotopy is lifted.

e-Open and e-Closed Spaces

If X and Y are spaces, we use the standard notation X^Y for the set of maps from Y to X. X^Y becomes a topological space under the compact-open topology, which coincides with the topology of uniform convergence (with respect to any fixed metric on X, or with respect to the unique uniform structure on X compatible with its topology). X^Y is metrizable.

If (Y, Z) is a space pair, for any X there is a natural restriction map from X^Y to X^Z, which is continuous. Denote by $X^{Z\uparrow Y}$ the range of this restriction map, i.e. the set of maps from Z to X which extend to a map from Y to X (we simply say such a map *extends to* Y). Examples show that $X^{Z\uparrow Y}$ is neither open nor closed in X^Z in general. We seek conditions on X insuring that $X^{Z\uparrow Y}$ is always open or closed in X^Z for any space pair (Y, Z).

Definition:

Let X be a space.

(i) X is *e-open* if, for every space pair (Y, Z), the set $X^{Z \uparrow Y}$ is open in X^Z.

(ii) X is *e-closed* if, for every space pair (Y, Z), the set $X^{Z \uparrow Y}$ is closed in X^Z.

There is a corresponding notion for C*-algebras (ℓ-open, ℓ-closed).

Example

The space N is e-closed but not e-open.

To show N is not e-open, take $Y = [0, 1]$ and $Z = N$. Define $\phi_n : N \to N$ by

$$\phi_n\left(\frac{1}{k}\right) = \frac{1}{k+n}$$

and $\phi_n(0) = 0$. Then no ϕ_n is extendible to Y. But (ϕ_n) converges uniformly to a constant function, which obviously extends.

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The argument that N is e-closed is more complicated.

More Examples

(i) If X is a space with a path component which is not closed (e.g. the topologist's sine curve or a solenoid), then X is not e-open or e-closed. Thus $C(X)$ is not ℓ-open or ℓ-closed.

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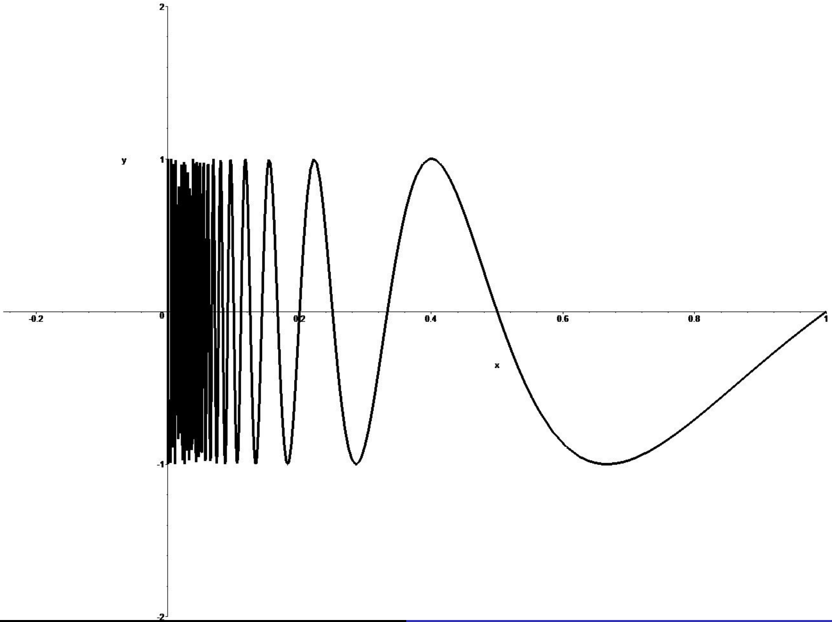
In both cases use $Y = [0, 1]$,

$$Z = N = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} .$$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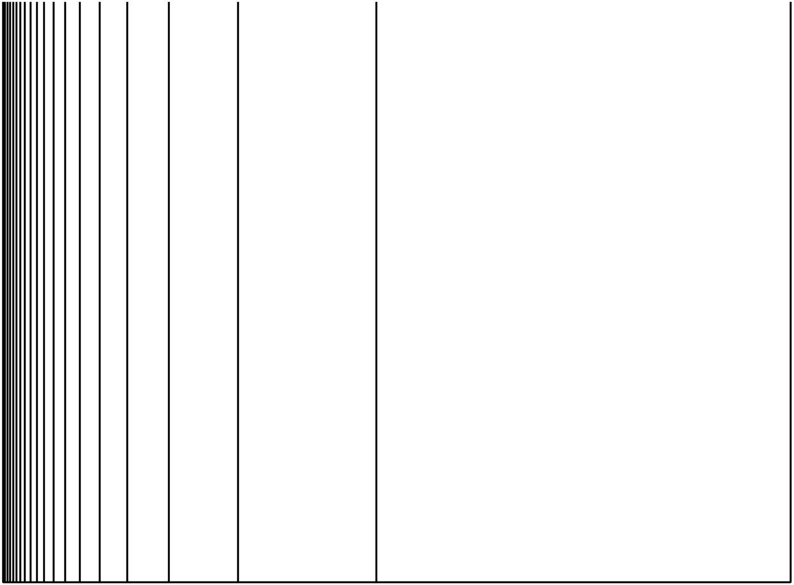

Easy corollary of the Close Lifting Theorem and its commutative counterpart:

Corollary:

(i) If A is a semiprojective C*-algebra, then A is both ℓ-open and ℓ-closed.

(ii) If X is an ANR, then X is both e-open and e-closed.

The converse is true in the commutative case for finite-dimensional spaces:

Theorem:

A finite-dimensional *e*-open space is an ANR.

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A finite-dimensional e-open space is an ANR.

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So it is reasonable to conjecture:

Conjecture:

Every e-open space is an ANR. Every ℓ-open C*-algebra is semiprojective.

The *e*-closed property is more mysterious.

Although there is no obvious direct proof that an *e*-open space is *e*-closed, I do not know an example of a space which is *e*-open but not *e*-closed), and I conjecture that none exist. There are *e*-closed spaces which are not *e*-open.

I do not have a good idea how to characterize *e*-closed spaces.