What Does “Obvious” Mean in Mathematics? (and why do we have to prove things?)

Bruce Blackadar

January 2015

" 'Obvious' is the most dangerous word in mathematics."

E. T. Bell

"Since people have tried to prove obvious propositions, they have discovered that many of them are false."

Bertrand Russell

"A simple man believes every word he hears; a clever man understands the need for proof."

Proverbs 14:15

I. WHAT IS OBVIOUS?

Areas in the Plane

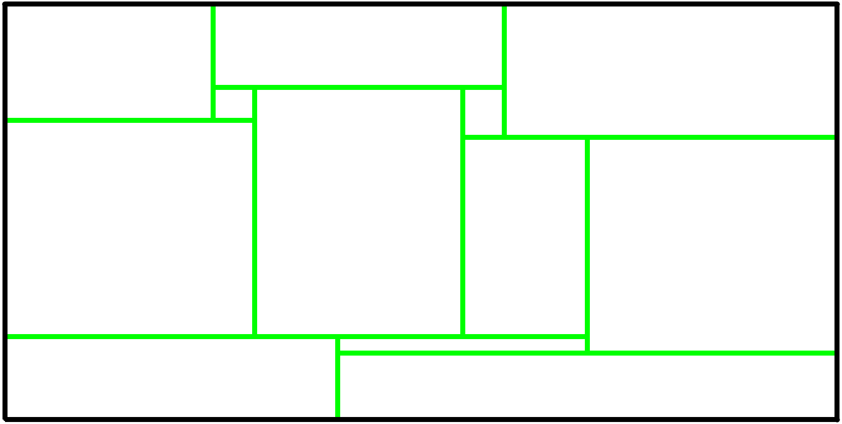
Statement 1.

If R is a rectangle with length x and width y, then the area $A(R)$ is xy.

Statement 2.

If a rectangle R is subdivided into a finite number $R_1, \ldots, R_n$ of nonoverlapping subrectangles (nonoverlapping means two of the subrectangles can have only boundary points in common), then

$$A(R) = \sum_{k=1}^{n} A(R_k) \ .$$



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Is Statement 2 obvious?

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The real question is:

Question:

Is Statement 1 a definition or a proposition?

If it is a proposition, what is the definition of the area of a rectangle?

Properties of Area

1. If R is a rectangle with length x and width y, then $A(R)$ is xy.
2. If X and Y are congruent, then $A(X) = A(Y)$.
3. If $X \subseteq Y$, then $A(X) \leq A(Y)$.
4. If X is subdivided into a finite number $X_1, \ldots, X_n$ of nonoverlapping pieces, then

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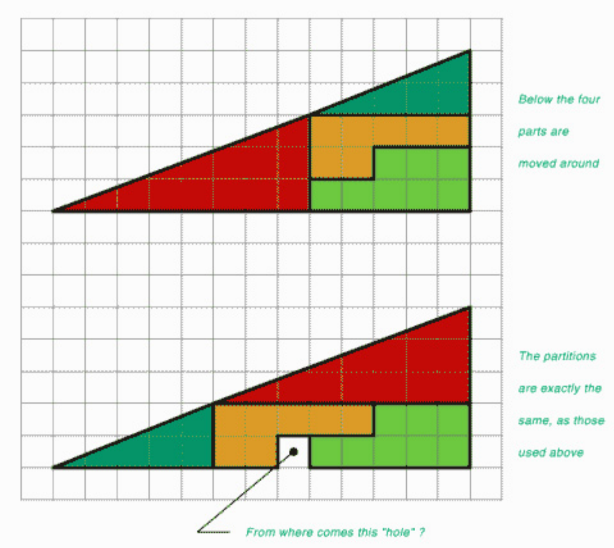
Question:

Is there a notion of area for plane figures with these properties?



A triangle of area $\frac{65}{2}$ is cut up and rearranged into a figure of area $\frac{63}{2}$ (or is it?)

Best solution: Take Statement 1 as the *definition* of the area of a rectangle.

We can then define the area of other subsets of the plane by covering them with rectangles and taking a limit (Lebesgue measure; Jordan content). Can be done somewhat more simply for polygons, but still takes considerable work.

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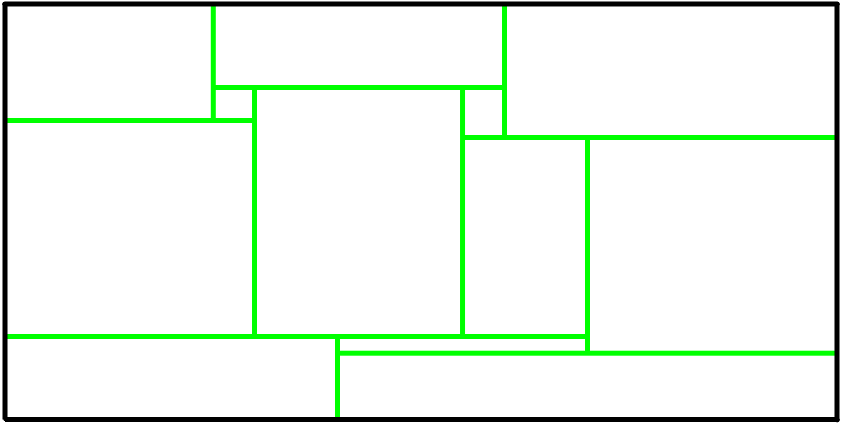
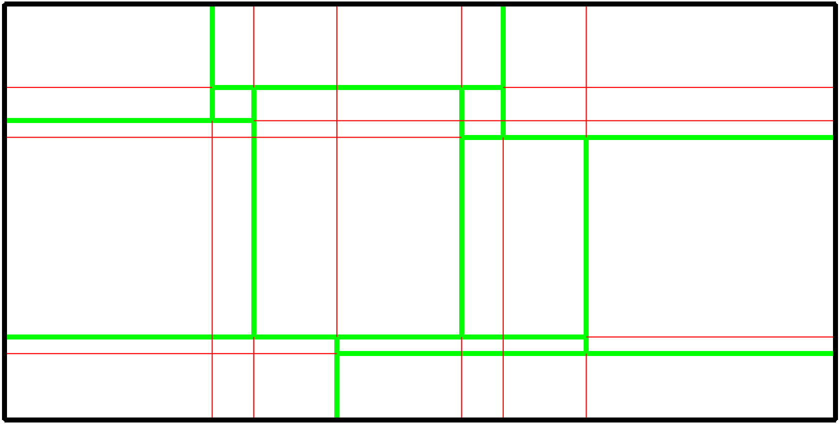
But property 4 is then not obvious, and must be proved, even for rectangles! (Statement 2)

The proof is nontrivial (but not hard). It cannot be proved by induction!





Apportionment

How do we apportion seats in the U.S. House of Representatives?

The Constitution requires that

> *"Representatives . . . shall be apportioned among the several States . . . according to their respective Numbers."*

It also requires that a census be taken every 10 years, and that the House be reapportioned as a result of the census. It does not say *how* this is to be done.

It seems obvious in principle: if we fix the size of the House at 435, each district should have a population of

$$p = \frac{\text{(Total U.S. population)}}{435}$$

and then a state should have a quota of

$$q = \frac{\text{(Total population of the state)}}{p} = \frac{\text{(State population)}}{\text{(U.S. population)}} \cdot 435$$

representatives.

So what's the problem? It is that q is rarely an integer, and the numbers need to be rounded to integers. The rounded numbers rarely add to 435. So it is a number roundoff problem.

There is a long and fascinating history of the attempts to solve the problem, both mathematically and politically.

Obvious properties that an apportionment scheme should satisfy:

1. [**Quota**] Every state's allocation should be one of the integers adjacent to its quota.
2. [**Population Monotonicity**] If a state's population increases, and other states' populations remain the same, the state's allocation should not decrease.
3. [**House Monotonicity**] If the size of the house is increased, no state's allocation should decrease.

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Theorem:

There is no apportionment method which is population monotone and always stays within the quota.

So there is no perfect system (or even indisputably best system)!

Calculus

It is surprising how many people think that analysis consists in the difficult proofs of obvious theorems. All we need know, they say, is what a limit is, the definition of continuity and the definition of the derivative. All the rest is 'intuitively clear.' "

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Theorem:

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Theorem:

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. If f' is identically 0, then f is constant.

Requires completeness of $\mathbb{R}$.

Suppose our number system is $\mathbb{Q}$. Define

$$f(x) = \begin{cases} 0 & \text{if} \quad x^2 < 2 \\ 1 & \text{if} \quad x^2 > 2 \end{cases} .$$

Then f is differentiable everywhere and $f' \equiv 0$, but f is not constant.

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Whitney's Example

There is a $\mathcal{C}^1$ surface in $\mathbb{R}^3$ (graph of a $\mathcal{C}^1$ function $f : \mathbb{R}^2 \to \mathbb{R}$) and a curve on the surface such that

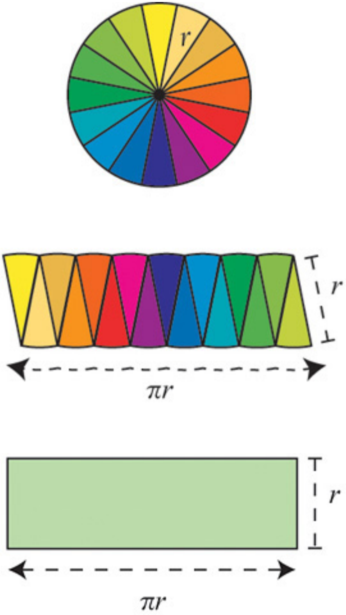
1. The tangent plane at every point of the curve is horizontal yet
2. The curve is not at constant height.

The curve is highly nondifferentiable (Hausdorff dimension 2). Impossible if f is $\mathcal{C}^2$ by Sard's Theorem.

Area of a Circle

ARCHIMEDES gave a beautiful argument indicating that the area of a circle (disk) of radius r is πr^2.

This argument has recently been publicized by S. STROGATZ, and is sometimes represented as a "proof" of the formula.


r
r
πr
r
πr

But is it really a proof? Several things need to be shown:

1. The circle can be evenly divided into such wedges (discussed later).
2. The number π is well defined (discussed later).
3. It must be shown that area is well defined for the types of regions considered, and that it has the subdivision property discussed earlier.
4. The limiting operations must be made precise.
5. It must be shown that both area and arc length are preserved under the limiting operations of 4.

Showing 5. is the trickiest part. Arc length is *not* preserved under uniform limits!

Simplest example: staircase approximating the diagonal of a square.

Another dramatic example: $f_n(x) = 2^{-n} \sin(4^n x)$, $0 \leq x \leq 2\pi$.
$f_n \to 0$ uniformly, but the arc lengths of the graphs go to infinity.

Good news: 5. is true: the derivatives also go uniformly to zero. So ARCHIMEDES' argument *can* be made rigorous (with more work).

"The kind of knowledge which is supported only by observations and is not yet proved must be carefully distinguished from the truth; it is gained by induction, as we usually say. Yet we have seen cases in which mere induction led to error. Therefore, we should take great care not to accept as true such properties of the numbers which we have discovered by observation and which are supported by induction alone. Indeed, we should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful."

L. Euler

II. HIDDEN ASSUMPTIONS

EUCLID'S geometry is traditionally regarded as a model of rigor. But it has shortcomings:

i. The postulates are vague and ambiguous, and open to interpretation.
ii. There are many hidden assumptions in EUCLID'S proofs which do not follow from the postulates.

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With suitable interpretation, spherical geometry does arguably satisfy EUCLID'S postulates.

Euclid's Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Ambiguities in the Postulates

1. Is the straight line in 1 unique? (Can two distinct lines have two points in common?)

> Spherical geometry satisfies Postulate 1 (without the uniqueness). Uniqueness holds on the projective plane.

2a. Is the extension in 2 unique? (Can two lines have a segment in common?)

2b. In the extension of the line segment from p to q to a line, is the extension beyond p disjoint from the extension beyond q? ("Infinitude of lines")

> Spherical Geometry satisfies Postulate 2 (with the uniqueness but without the "infinitude of lines").

3. In 3, what does "distance" mean?

To EUCLID, a "distance" would have meant the "magnitude" of a line segment between two points. Thus he arguably would have agreed that Postulate 3 could have been equivalently phrased:

3′. Given points p and q, a circle can be drawn with center p passing through q.

In fact, this is the form of Postulate 3 always used in his proofs.

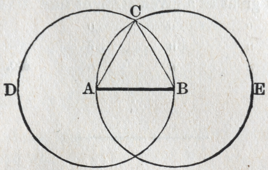
Spherical Geometry satisfies Postulate 3′.

4. Postulate 4 only makes sense if there is an assumed set (group) of sufficiently many rigid motions available to translate and rotate figures.

Spherical Geometry satisfies Postulates 4 and 5.

PROPOSITION 1. PROBLEM.

To describe an equilateral triangle on a given straight line.



Let AB be the given straight line :
it is required to describe an equilateral triangle on ***AB.***

With centre A and radius AB, describe ⊙ BCD. *Post.* 3
With centre B and radius BA, describe ⊙ ACE; *Post.* 3
and let the two circles intersect at C.
Join AC, BC. *Post.* 1

ABC shall be an equilateral triangle.

For $AB = AC$, being radii of the ⊙ BCD; *I. Def.* 16
and $AB = BC$, being radii of the ⊙ ACE; *I. Def.* 16
$\therefore$ $AC = BC$. *I. Ax.* 1
$\therefore$ AB, AC, BC are all equal,
and ABC is an equilateral triangle. *I. Def.* 23

How Do We Define π?

It is usually defined as the ratio of the circumference of a circle to its diameter (for this to make sense, it must be shown that the ratio is always the same for all circles, which is not entirely obvious).

Several things must be made precise before this can be accepted as a real definition: the terms *circumference*, *circle*, and *diameter* must be defined.

The real issue is the definition of the term *circumference*. Although people have a good intuition as to what this term means, and one could be led to say the meaning is "obvious", it is not at all easy to give a precise definition, and the meaning of the word is finessed throughout precalculus mathematics.

Using calculus, it can be defined as a special case of arc length, which can itself be defined as the supremum of a set of sums (and in good cases exactly calculated as a definite integral). It is then not entirely obvious that a circle even has a (finite) circumference.

This was more or less shown already in ancient times by a limit argument, using inscribed and circumscribed regular polygons, and the numerical value of π was calculated to increasing accuracy over the centuries.

But making good precise sense of the circumference of a circle, and hence the number π, is a quite subtle matter.

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Defining surface area of a surface is even much trickier than defining arc length of a curve!

How do you prove that there exist regular n-gons in the plane (without using angle measure)?

This cannot be proved from Euclid's postulates: in the geometry consisting of constructible points, there are only regular n-gons for certain n, not all n.

Simplest argument(?): there is an inscribed n-gon in the unit circle of maximum perimeter by compactness. It is easy to show that an inscribed n-gon of maximum perimeter is regular.

There is also a purely algebraic proof (not using completeness of $\mathbb{R}$).

The Axiom of Choice

Axiom of Choice:

From any collection of nonempty sets, an element can be chosen from each set.

This seems like a no-brainer, and mathematicians used it almost without notice until the early 20th century when it was discovered not to be obvious after all! Now some mathematicians refuse to use it.

As an analyst, I like the AC: many things in analysis depend on it for proof.

> *"Assuming the AC can do no mathematical harm that has not already been done."*

R. Boas

The following depend on the Countable AC. Analysis in its usual form would be impossible without them.

1. Every infinite set contains a sequence of distinct elements.
2. The sequential criterion for continuity of functions from $\mathbb{R}$ to $\mathbb{R}$ is equivalent to the $\epsilon - \delta$ definition.
3. $\mathbb{R}$ is not a countable union of countable sets.

Euler's Formula

For any polyhedron in $\mathbb{R}^3$, we have

$$V - E + F = 2 \ .$$

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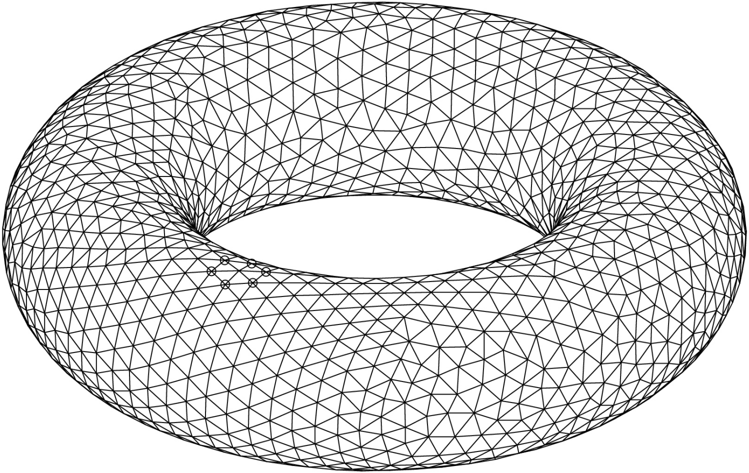
Is this formula true for *any* polyhedron? (And what is the right definition of a polyhedron?)

Yes, for convex polyhedra.

No, for torus-like polyhedra.

A precise, correct statement is tricky to make. (Leads to *Euler characteristic*).

See Lakatos, *Proofs and Refutations*



Triangulations

Theorem:

Every closed surface is homeomorphic to a polyhedron (can be *triangulated*).

What about higher dimensions:

Questions:

1. Can every closed manifold be triangulated?
2. [**Hauptvermutung**] Are any two triangulations of a closed manifold equivalent (do they have a common refinement)?

Answers yes for 3-manifolds, yes for all smooth manifolds, but no for topological 4-manifolds!

III. DEFINITIONS AND THEOREMS MUST BE CAREFULLY STATED (AND APPLIED!)

L'Hôpital's Rule

Theorem: (Informal)

Let $\lim_{x \longrightarrow a} \frac{f(x)}{g(x)}$ be an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = \lim_{x \longrightarrow a} \frac{f'(x)}{g'(x)} .$$

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Let $f(x) = x^2 \sin \frac{1}{x}$, $g(x) = x$. $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$.

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$$

does not exist.

Example (Stolz, 1879):

Let $f(x) = x + \sin x \cos x$. Then $f'(x) = 2\cos^2 x$.

Let $g(x) = e^{\sin x} f(x)$. $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = +\infty$

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$$\lim_{x \to +\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to +\infty} \frac{2\cos^2 x}{e^{\sin x}[f(x)\cos x + 2\cos^2 x]}$$

$$= \lim_{x \to +\infty} \frac{2\cos x}{e^{\sin x}[f(x) + 2\cos x]} = 0 \ .$$

Formal Manipulation of Power Series

"A power series is, after all, a rather long polynomial."

Forman S. Acton

To solve the differential equation (IVP)

$$y' = f(t, y), \quad y(0) = 0$$

where f is analytic ("engineer's solution"):

1. Write the solution as $\sum_{n=1}^{\infty} a_n t^n$ with unknown a_n.
2. Substitute into the power series for f and collect terms.
3. Solve equations successively for the a_n.

To get a numerical approximation to the solution, only need a few a_n.

Theorem(?):

If $x \neq 0$ or 1, then $\displaystyle\sum_{n=-\infty}^{+\infty} x^n = 0$.

Proof: We have

$$\sum_{n=-\infty}^{+\infty} x^n = \sum_{n=1}^{+\infty} x^n + \sum_{n=0}^{+\infty} x^{-n}$$

$$= \sum_{n=1}^{+\infty} x^n + \sum_{n=0}^{+\infty} \left(\frac{1}{x}\right)^n$$

$$= \frac{x}{1-x} + \frac{1}{1-\frac{1}{x}} = \frac{x}{1-x} + \frac{x}{x-1} = 0 \ .$$

Convergence of Functions

Consider the following two questions:

Question:

If $f_n \longrightarrow f$ and each f_n is continuous, is f continuous?

Question:

If $f_n \longrightarrow f$, does $\int_a^b f_n(x)\,dx \ \longrightarrow \int_a^b f(x)\,dx$?

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We understand today that we must first precisely answer

Question:

What do we mean when we say $f_n \longrightarrow f$?

It is easy to make various precise statements about both questions with different hypotheses on the type of convergence, some of which are true and some false.

It is not so easy to make a precise statement (precise hypotheses) such that

1. The statement is actually true.
2. The hypotheses are satisfied in many, if not all, applications of interest.

There are such general statements (e.g. Monotone and Dominated Convergence Theorems).

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Every map can be colored with four colors.

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This statement is not precise. To make a precise statement, we need to make some definitions.

First try:

A *map* is a division of the plane into a finite number of nonoverlapping (connected) regions called *countries*.

Two countries are *adjacent* if they share a common piece of boundary (more than just isolated points).

A *coloring* of a map is an assignment of colors so that adjacent countries have different colors.

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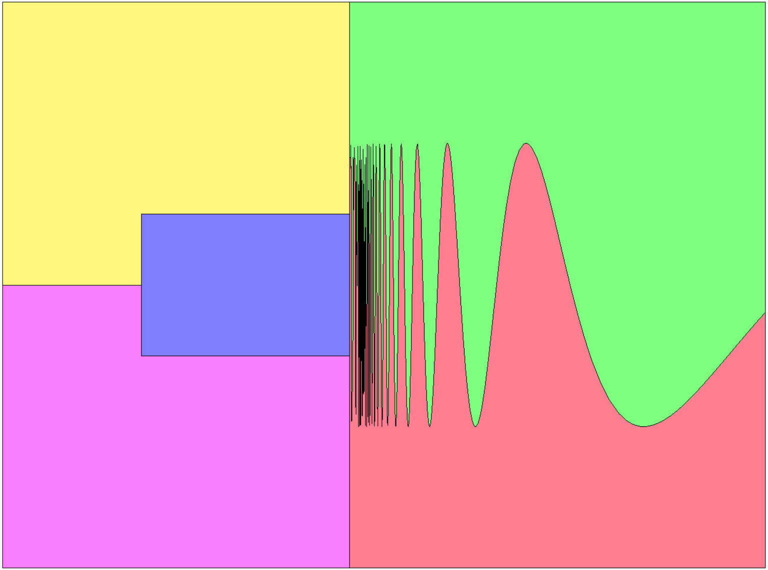
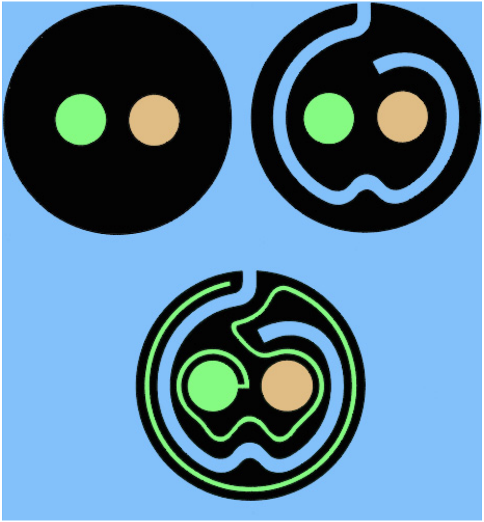
Are these definitions adequate?



The Lakes of Wada



"After deserting for a time the old Euclidean standards of rigour, mathematics is now returning to them, and even making efforts to go beyond them. ... Later developments ... have shown more and more clearly that in mathematics a mere moral conviction, supported by a mass of successful applications, is not good enough. Proof is now demanded of many things that formerly passed as self-evident. Again and again the limits to the validity of a proposition have been in this way established for the first time. ... In all directions these same ideals can be seen at work – rigour of proof, precise delimitation of extent of validity, and as a means to this, sharp definition of concepts."

Gottlob Frege, 1884

IV. CONCLUSION

Moral

1. The mathematical interpretations of a statement are often vast, far beyond one's initial expectation. It is unrealistic to expect to foresee all the possibilities in many cases. There can be unforeseen cases where the statement is false.

2. As a result, it is vital to prove mathematical statements using strict logical arguments from agreed basic axioms without making additional assumptions, no matter how "obvious" they seem. This is the only way to know that a statement is really *always* true.

3. It is not possible to prove everything in mathematics; we have to start somewhere. But the beginning assumptions must be clearly stated, and must be regularly reexamined to see if they are indeed a reasonable starting point for the theory.

Advice

1. Do not be gun-shy: believe in your intuition. But don't entirely trust it; do not fully accept a conclusion unless it is proved.

2. If an instructor (or colleague) ever tells you something is obvious and you aren't so sure, or it is not obvious to you, don't be reluctant or embarrassed to ask for an explanation. It may turn out that it is not so obvious to the instructor either!

"Trust but verify."

"Proclus shared the human problem of distinguishing between the obvious and that which only appears to be obvious."

George E. Martin