Introduction to Noncommutative Topology

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Topology

Three flavors of topology:

General (point-set) topology

Algebraic topology

"Combinatorial topology"

We will mostly only consider topology of compact Hausdorff spaces.

Algebraic Approach to Topology

Let X be a topological (compact Hausdorff) space.

Idea: Study X by studying real- or complex-valued continuous functions on X. (Basic philosophy of Functional Analysis)

Let $C(X)$ be the set of complex-valued continuous functions on X. Study X by studying $C(X)$.

$C(X)$ is a unital ring (algebra over $\mathbb{C}$). Theme: Algebraic properties of $C(X)$ encode the space X and its topology.

Recovering X from $C(X)$

Common construction in mathematics (algebraic geometry): If R is a ring, associate to R a topological space of ideals of R. Technicality: must use prime ideals.

Definition:

If R is a unital ring, let $\mathrm{Maxspec}(R)$ be the set of maximal ideals of R.

Proposition:

The maximal ideals of $C(X)$ are in one-one correspondence with the points of X:

$$p \in X \leftrightarrow M_p = \{f \in C(X) | f(p) = 0\}$$

Recovering the Topology

If R is a unital ring, put a topology on $\mathrm{Maxspec}(R)$ (hull-kernel topology):
If $S = \{M_\lambda | \lambda \in \Lambda\}$ is a set of maximal ideals in R, set

$$\bar{S} = \{M \in \mathrm{Maxspec}(R) | \cap_\lambda M_\lambda \subseteq M\} \ .$$

Theorem:

Let X be a compact Hausdorff space. If $\mathrm{Maxspec}(C(X))$ is identified with X via $p \leftrightarrow M_p$, the hull-kernel topology on $\mathrm{Maxspec}(C(X))$ is precisely the original topology on X.

Another Way to Recover X

Each $p \in X$ corresponds to an algebra-homomorphism ϕ_p from $C(X)$ to $\mathbb{C}$:

$$\phi_p(f) = f(p)$$

These are all the algebra-homomorphisms from $C(X)$ to $\mathbb{C}$.

The kernel of ϕ_p is M_p.

Theorem:

Let $\widehat{C(X)} = \{\phi_p | p \in X\}$. Identify $\widehat{C(X)}$ with X and $\mathrm{Maxspec}(C(X))$ by $p \leftrightarrow \phi_p \leftrightarrow M_p$. Then the weak-* topology (topology of pointwise convergence) coincides with the hull-kernel topology on $\mathrm{Maxspec}(C(X))$ and the original topology on X.

C*-Algebras

Definition:

A C*-algebra is a complex Banach algebra A, with an involution * satisfying

$$(x+y)^* = x^* + y^*\ ,\ (\lambda x)^* = \bar{\lambda} x^*\ ,\ (xy)^* = y^* x^*\ ,\ (x^*)^* = x$$

for all $x, y \in A, \lambda \in \mathbb{C}$, and satisfying the C*-axiom

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in A\ .$$

Ingredients:

(i) Algebraic structure: complex vector space, ring, adjoint

(ii) Topological structure: norm

Standard Examples

1. $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, or any norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ (*concrete C*-algebra*). In particular, $\mathbb{M}_n = M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$.

2. $C(X)$, X a compact Hausdorff space. The norm is the supremum norm, and the adjoint is complex conjugation.

Theorem (Gelfand-Naimark 1943):

(a) Every C*-algebra is isometrically *-isomorphic to a norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

(b) Every unital commutative C*-algebra is isometrically *-isomorphic to $C(X)$ for some compact Hausdorff space X.

C*-Algebras are not necessarily unital. In this talk we will restrict attention to unital C*-algebras, and all homomorphisms (*-homomorphisms) will be assumed to be unital too. Such homomorphisms are automatically norm-decreasing, and isomorphisms are automatically isometric.

Thus C*-algebras are "noncommutative topological spaces."

The correspondence between X and $C(X)$ goes much deeper:

Theorem (Gelfand):

The correspondence $X \leftrightarrow C(X)$ is a contravariant category equivalence between the category of compact Hausdorff spaces and continuous maps and the category of unital commutative C*-algebras and (unital) *-homomorphisms.

This means also: if $f : X \to Y$ is a continuous function, then there is a corresponding unital *-homomorphism $\tilde{f}$ from $C(Y)$ to $C(X)$, with nice properties, and every unital *-homomorphism is of this form. For one direction, $\tilde{f}(g) = g \circ f$; for the other, if $\psi : C(Y) \to C(X)$ is a *-homomorphism, and $p \in X$, then $\phi_p \circ \psi$ is a homomorphism from $C(Y)$ to $\mathbb{C}$, hence of the form ϕ_q for some $q \in Y$; set $f(p) = q$. Check that f is continuous and $\tilde{f} = \psi$.

Thus, in principle, $C(X)$ encodes the entire topological structure of X, and all of topology (of compact Hausdorff spaces) can be done on the C*-algebra level.

Admission: This is *not* the way to do most of topology!

However, some topological notions, e.g. K-theory, are done very efficiently at the C*-algebra level and provide important tools in the theory of Operator Algebras.

The interaction goes both ways: Operator Algebras have nice applications in topology and geometry.

Noncommutative Homotopy and Shape Theory

If X and Y are spaces and f_0, f_1 are continuous functions from X to Y, then f_0 and f_1 are *homotopic* if there is a continuous function $h : [0,1] \times X \to Y$ such that $h(0,x) = f_0(x)$ and $h(1,x) = f_1(x)$ for all $x \in X$. If $f_t : X \to Y$ is defined by $f_t(x) = h(t,x)$, then the f_t give a "continuous deformation" from f_0 to f_1.

Exact C*-analog: If α_0, α_1 are *-homomorphisms from a C*-algebra A to a C*-algebra B, a *homotopy* from α_0 to α_1 is a *-homomorphism $\beta : A \to C([0,1], B)$ with $\phi_0 \circ \beta = \alpha_0$ and $\phi_1 \circ \beta = \alpha_1$, where $\phi_t : C([0,1], B) \to B$ is evaluation at t.

Can do homotopy theory and shape theory (generalized homotopy theory) for C*-algebras.

Interlude: The Stone-Čech Compactification

If X is a completely regular topological space, then X can be embedded as a dense subspace of a compact Hausdorff space Y in a maximal way. Y is called the *Stone-Čech compactification* of X, denoted βX.

Here is a very elegant construction of βX: let $C_b(X)$ be the set of bounded complex-valued continuous functions on X. With the usual operations and supremum norm, $C_b(X)$ is a commutative C*-algebra, hence of the form $C(Y)$ for some compact Hausdorff space Y. X embeds in Y since point evaluation at any point of X is a homomorphism from $C_b(X)$ to $\mathbb{C}$. It is straightforward to show that Y has the right universal property, hence $Y \cong \beta X$.

Noncommutative version: multiplier algebra of a C*-algebra.

Applications

Traditional:

Group representations
Mathematical physics (quantum mechanics)
Dynamical systems

Modern:

Topology and geometry
Free probability
ETC.

Key Application: Dynamical Systems

A *(topological) dynamical system* is an action of a group G (e.g. $\mathbb{Z}$, $\mathbb{R}$) on a topological space X by homeomorphisms.
If G is a topological group (e.g. $\mathbb{R}$), need a continuity condition.
We will restrict to the case where G and X are locally compact.
If G acts on X by homeomorphisms, it acts on $C_0(X)$ by automorphisms (and conversely).

Definition:

A *C*-dynamical system* (A, G, α) is a (continuous) action α of a locally compact topological group G on a C*-algebra A, i.e. a (continuous) homomorphism from G into $\mathrm{Aut}(A)$.

If (A, G, α) is a C*-dynamical system, can construct a *crossed product C*-algebra* $A \rtimes_\alpha G$. Roughly speaking, it contains A as a C*-subalgebra and a group of unitary elements isomorphic to G which conjugate A by α.

In particular, if G acts on X, can form the *transformation group C*-algebra* $C(X) \rtimes_\alpha G$.

The crossed product is an important tool in studying dynamical systems.

Example: Putnam *et al.* used C*-algebras to classify minimal homeomorphisms ($\mathbb{Z}$-actions) of the Cantor set, critical to symbolic dynamics.

More generally, a C*-algebra can be constructed from certain groupoids (Kumjian).

Group C*-Algebras

Important special case: X is a single point, G acting trivially. The crossed product is called the *group C*-algebra* $C^*(G)$.

The group C*-algebra has long been a key tool in studying unitary representations of locally compact groups.

Harmonic Analysis and Pontrjagin Duality

Harmonic analysis = analysis on a locally compact abelian group G (e.g. $\mathbb{T}$, $\mathbb{R}$) via Fourier transform to the Pontrjagin dual group $\hat{G}$.

Connection with group C*-algebras: if G is a locally compact abelian group, then $C^*(G)$ is a commutative C*-algebra, hence of the form $C_0(Y)$ for some locally compact Hausdorff space Y. It turns out that Y is $\hat{G}$, and the identification of $C^*(G)$ with $C_0(\hat{G})$ is just the Fourier transform.

Topological K-Theory

(Complex) K-theory is the theory of (complex) vector bundles. If X is a compact Hausdorff space, the isomorphism classes of (complex) vector bundles over X form an abelian semigroup $V(X)$ under Whitney sum.

Definition:

$K^0(X)$ is the Grothendieck group (group of formal differences) of $V(X)$.

Can extend to locally compact spaces. Define $K^1(X) = K^0(SX)$, where $SX = X \times \mathbb{R}$. Inductively define $K^{n+1}(X) = K^n(SX)$.

Some of the key facts about (complex) topological K-theory:

Theorem:

(i) K-Theory is an extraordinary cohomology theory on (locally) compact Hausdorff spaces.

(ii) (**Bott Periodicity**) $K^2(X) = K^1(SX) = K^0(S^2X) \cong K^0(X)$ for any X.

(iii) (**Chern Character**) If X is a compact Hausdorff space, then

$$K^0(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ even}} H^n(X, \mathbb{Q})$$

$$K^1(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ odd}} H^n(X, \mathbb{Q})$$

K_0-**Theory**

K-Theory can be done very nicely at the C*-algebra level. Begin with K^0, which works for general rings.

Let A be a unital C*-algebra (or unital ring). Consider projections (idempotents) in matrix algebras over A. If p is a projection in $M_n(A)$, for $m > n$ identify p with the projection $diag(p, 0)$ in $M_m(A)$. Consider the equivalence relation on projections in matrix algebras over A generated by this identification and ordinary equivalence. There is a natural sum on equivalence classes: $[p] + [q] = [diag(p, q)]$ (this is well defined!) making the set $V(A)$ of equivalence classes into an abelian semigroup with identity [0].

Let $K_0(A)$ be the Grothendieck group of $V(A)$. (Subscript since it is covariant!)

Proposition:

If X is a compact Hausdorff space, then $K_0(C(X))$ is naturally isomorphic to $K^0(X)$.

K_0 of a nonunital C*-algebra (nonunital ring) can also be defined.

Topological K^1 also has a construction on the C*-algebra level (not in general rings!):

Definition:

Let A be a unital C*-algebra. Then

$$K_1(A) = \lim_{\rightarrow} U(M_n(A))/U(M_n(A))_0 = \lim_{\rightarrow} GL_n(A)/GL_n(A)_0 \ .$$

We have $K_1(C(X)) \cong K^1(X)$ (not as obvious).

K-Theory has become one of the most fundamental tools in the study of C*-algebras. One important feature of C*-algebra K-theory, not widely used in topological K-theory, is a partial order on K_0 (taking the image of $V(A)$ as positive cone).

There has recently been a complete classification of a large class of simple C*-algebras, essentially via K-theory.

Dimension Theory

How do we define the dimension of a topological space? This is difficult, and there are many ways to do it:

- Covering dimension
- Small and large inductive dimension
- . . .

A remarkable theorem says that all these dimension theories agree (give the same numerical dimension) on all separable metrizable spaces, in particular all compact metrizable spaces, and the dimension of $\mathbb{R}^n$ is n.

Covering dimension can be rephrased in algebra language in several different ways, giving various definitions of “dimension” for C*-algebras:

- Stable rank
- Real rank
- Nuclear dimension
- ...

These do *not* agree even on elementary examples of C*-algebras, and each gives different structural information on the algebra.

Nuclear dimension turned out to be the last main piece of the story in the classification program.

Combinatorial (PL) Topology

A *triangulation* of a (compact metrizable) space X is a homeomorphism from X onto a simplicial complex K.

Idea: If X is triangulated by K, the topology of X is encoded in the combinatorics of K.

PL Topology: Only consider maps between spaces which are piecewise linear (affine) with respect to some triangulations of the spaces.

For the general setting, we consider the following fact:

Theorem:

Every compact metrizable space can be written as an inverse limit of a sequence of simplicial complexes with (surjective) piecewise-linear connecting maps.

$$X \to \cdots \to K_3 \to K_2 \to K_1$$

In fact, the simplicial complexes can be subdivided so that the continuous functions from X to $\mathbb{C}$ of the form $f \circ \phi_n$, where $\phi_n : X \to K_n$ is the connecting map and $f : K_n \to \mathbb{C}$ is piecewise-linear, are dense in $C(X)$.

This construction is used e.g. in defining Čech cohomology.

Definition:

A *weak triangulation* of a compact metrizable space X is a continuous map from X to a simplicial complex K.

The theorem can be interpreted to say that a general compact metrizable space can be "approximately triangulated," i.e. arbitrarily approximated by weak triangulations.

There is a (weak) sense in which the approximation is unique (related to shape theory).

Algebraic Version of Weak Triangulation

If X is a compact metrizable space, a weak triangulation of X by a simplicial complex K with n vertices is effectively the same thing as a unital completely positive map from $\mathbb{C}^n$ to $C(X)$. If $f : X \to K$ is the weak triangulation, the image of $\mathbb{C}^n$ in $C(X)$ is the set of continuous functions from X to $\mathbb{C}$ of the form $g \circ f$, where $g : K \to \mathbb{C}$ is piecewise-linear.

The theorem can then be restated:

Theorem:

Let X be a compact metrizable space. Then there is a coherent sequence of unital completely positive maps $\phi_k : \mathbb{C}^{n_k} \to C(X)$ and $\phi_{k,j} : \mathbb{C}^{n_k} \to \mathbb{C}^{n_j}$ $(k < j)$ such that the $\phi_{k,j}$ are "asymptotically multiplicative" and $\cup_k \phi_k(\mathbb{C}^{n_k})$ is dense in $C(X)$.

$$\mathbb{C}^{n_1} \to \mathbb{C}^{n_2} \to \cdots \to C(X)$$

Thus $C(X)$ is an "inductive limit" of finite-dimensional commutative C*-algebras with unital completely positive, asymptotically multiplicative connecting maps.

E. Kirchberg and I studied a noncommutative version:
When can a (separable, unital) C*-algebra A be written as an "inductive limit" of finite-dimensional C*-algebras with unital completely positive asymptotically multiplicative connecting maps? We call such a C*-algebra an *NF algebra*.

Fact: A finite-dimensional C*-algebra is a finite direct sum of matrix algebras over $\mathbb{C}$.

Theorem (B.-Kirchberg):

A separable (unital) C*-algebra is an NF algebra if and only if it is nuclear and quasidiagonal.

Conjecture:

A separable C*-algebra is an NF algebra if and only if it is nuclear and stably finite.

Quasidiagonality, although still not well enough understood, plays a crucial role in the classification program.