

SCALING ELEMENTS REVISITED

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Joint work with J. Cuntz and A. Kumjian.

Theorem. Let A be a stable simple C^* -algebra. Then the following are equivalent:

- (i) A is algebraically simple.
- (ii) A contains an infinite projection.
- (iii) There is no finite nonzero dimension function on the Pedersen ideal $\mathcal{P}(A)$.
- (iv) There is no nonzero (necessarily faithful) densely defined quasitrace on A .

If A is exact, these are also equivalent to

- (v) There is no nonzero (necessarily faithful) densely defined trace on A .
- (vi) \tilde{A} contains an infinite projection.

SOME CONSEQUENCES

Corollary. Let A and B be simple C^* -algebras. If B contains an infinite projection, so does $A \otimes B \otimes \mathbb{K}$.

Definition. A C^* -algebra A is *completely projectionless* if $A \otimes B$ is projectionless for all C^* -algebras B .

Examples. (i) $C_0(X) \otimes B$, X connected and noncompact, B arbitrary

(ii) Any [sub]contractible C^* -algebra

Rigidity Property:

Corollary. A simple C^* -algebra cannot be written as an inductive limit of completely projectionless C^* -algebras.

Definition. Let x be an element of a C^* -algebra A . Then x is a *scaling element* if $xx^* \neq x^*x$ and $xx^* \ll x^*x$, i.e. $(x^*x)(xx^*) = xx^*$.

Simple functional calculus arguments show:

Proposition. Let A be a C^* -algebra. Then

- (i) If A contains a scaling element then $\mathcal{P}(A)$ contains a scaling element x and $a \geq 0$ such that $\|x\| = \|a\| = 1$ and $a \ll x^*x$, $a \perp xx^*$.
- (ii) If A contains an “approximate scaling element,” then A contains a scaling element.

Proposition. If A is a unital C^* -algebra containing a scaling element, then A contains an infinite projection.

Proof: If x is a scaling element in A as above, then $u = x + (1 - x^*x)^{1/2}$ is a nonunitary isometry.

A C^* -algebra A containing a scaling element contains a nonzero projection: if $x \in A$ is as above, and $u = x + (1 - x^*x)^{1/2} \in \tilde{A}$, then $1 - uu^*$ is a nonzero projection in A .

A C^* -algebra containing a scaling element need not contain an infinite projection.

Original proof of proposition showed:

If A contains a scaling element x , then there is a sequence (p_k) of mutually orthogonal, mutually equivalent nonzero projections in A , with $p_k \ll x^*x \forall k$.

Proposition. Let A be a simple C^* -algebra. If A contains a scaling element, then A contains an infinite projection.

Proof: Let x be a scaling element in $\mathcal{P}(A)$, and (p_k) a sequence of projections as above. Let $u_k^* u_k = p_1$ and $u_k u_k^* = p_k$ for each k . There are $y_1, \dots, y_n \in A$ with

$$\begin{aligned} x^* x &= \sum_{k=1}^n y_k^* p_1 y_k = \sum_{k=1}^n y_k^* u_k^* p_k u_k y_k \\ &= \left(\sum_{k=1}^n y_k^* u_k^* \right) \left(\sum_{k=1}^n p_k \right) \left(\sum_{k=1}^n u_k y_k \right) = c^* p c \end{aligned}$$

Set $q = p + p_{n+1} = \sum_{k=1}^{n+1} p_k$. Then

$$q = q(x^* x)q = q c^* p c q = u^* u \text{ for } u = p c q$$

and $u u^* \leq p$, so q is infinite.

Proof of Theorem

(ii) \Rightarrow (vi) and (ii) \Rightarrow (iii) \Rightarrow (iv) are trivial.

(iv) \Rightarrow (iii) is Handelman.

(v) \Rightarrow (iv) in exact case is Haagerup.

Add an additional condition:

(vii) A contains a scaling element.

Main proof: show (vii) \Rightarrow (ii) (above),

(i) \Leftrightarrow (vii), (iii) \Rightarrow (vii).

Proofs are similar to proof of (vii) \Rightarrow (ii).

(vii) \Rightarrow (i) is true in nonstable case too.

(vi) \Rightarrow (v): if \tilde{A} contains a nonunitary isometry, then it has one of the form $1 + y$ with $y \in \mathcal{P}(A)$; $y^*y - yy^* \not\cong 0$. If τ is a trace on $\mathcal{P}(A)$, then

$$\tau(yy^*) = \tau(y^*y) = \tau(yy^*) + \tau(y^*y - yy^*)$$

Hyponormal Elements

Definition. An element x in a C^* -algebra is *hyponormal* if $xx^* \leq x^*x$.

If A is stable, simple, and exact, the conditions of the theorem are equivalent to:

(viii) $\mathcal{P}(A)$ contains a nonnormal hyponormal element.

It is unclear whether this is equivalent to

(ix) A contains a nonnormal hyponormal element.

It is also unclear whether (viii) \Leftrightarrow (i)-(iv) [and also whether (vi) \Leftrightarrow (i)-(iv)] in the nonexact case.

Construction of Projectionless C*-Algebras

Let X be a closed orientable 3-manifold. $H^3(X) \cong \mathbb{Z}$.

Let $D(X, k)$ be the stable continuous trace C*-algebra with spectrum X and Dixmier-Douady invariant k .

$D(X, k)$ is projectionless if $k \neq 0$.

If σ is an orientation-preserving homeomorphism of X , then there is an automorphism α of $D(X, k)$ inducing σ , for any k .

Twisted Double Embedding (B.-Kumjian)

Embed $D(X, k)$ into $M_2(D(X, k)) \cong D(X, k)$ by $x \mapsto \text{diag}(x, \alpha(x))$.

Iterate the construction, and let $A(X, k, \alpha)$ be the inductive limit. Then $A(X, k, \alpha)$ is projectionless if $k \neq 0$.

Proposition. If σ is minimal, then $A(X, k, \alpha)$ is simple.

The isomorphism class of $A(X, k, \alpha)$ probably depends only on σ (and k).

It depends on more than the homotopy class of σ , which determines $K_*(A(X, k, \alpha))$: it also depends on the space of invariant measures, which determines the trace space.

Perhaps the Elliott invariant is a complete invariant. But:

The opposite algebra of $D(X, k)$ is isomorphic to the algebra $D(X, -k)$.

If X has no orientation-reversing homeomorphisms (e.g. certain lens spaces), and $k \neq 0$, then $D(X, k)$ is not anti-isomorphic to itself.

If such an X has a minimal homeomorphism σ , then $A(X, k, \alpha)$ is a candidate for a simple nuclear (stable, projectionless) C*-algebra not anti-isomorphic to itself.

The Elliott invariant does not distinguish between A and A^{op} .

Now let $X = \mathbb{T}^3$. Then $K_*(D(\mathbb{T}^3, k))$ can be calculated by realizing $D(\mathbb{T}^3, k)$ as a mapping torus of an automorphism of $C(\mathbb{T}^2) \otimes \mathbb{K}$: if $k \neq 0$, then

$$K_0(D(\mathbb{T}^3, k)) \cong \mathbb{Z}^3$$

$$K_1(D(\mathbb{T}^3, k)) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_k$$

If σ is a minimal homeomorphism (e.g. an irrational rotation), then $K_*(A(X, k, \alpha))$ can be computed.

Now tensor with O_∞ . From the Theorem, we obtain:

Corollary. $A(\mathbb{T}^3, k, \alpha) \otimes O_\infty$ contains an infinite projection. Thus $D(\mathbb{T}^3, k) \otimes O_\infty$ contains infinite projections.

In fact, a multiple of every element of

$$K_0(D(\mathbb{T}^3, k) \otimes O_\infty) \cong K_0(D(\mathbb{T}^3, k))$$

is representable by an infinite projection in the algebra $D(\mathbb{T}^3, k) \otimes O_\infty$.

If p is an infinite projection in $B = D(\mathbb{T}^3, k) \otimes O_\infty$, then $A = pBp$ is a locally trivial bundle algebra over \mathbb{T}^3 with fiber O_∞ (or something Morita equivalent to it.)

The bundle is nontrivial by K -theory.

One gets similar bundles over any closed orientable 3-manifold with a minimal homeomorphism (e.g. S^3).

Conclusion: There is interesting topology (nontrivial higher homotopy) in $Aut(O_\infty)$.

Question: What happens if use O_2 instead of O_∞ ? Are the bundles trivial? The K -theory is 0.