Some Exotic Manifolds:
Beyond Algebraic Topology

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"The element of play, which makes recreational mathematics recreational, may take many forms ... Perhaps this need for play is behind even pure mathematics. There is not much difference between the delight a novice experiences in cracking a clever brain teaser and the delight a mathematician experiences in mastering a more advanced problem. Both look on beauty bare – that clean, sharply defined, mysterious, entrancing order that underlies all structure. ... It is not surprising, therefore, that it is often difficult to distinguish pure from recreational mathematics. ... Creative mathematicians are seldom ashamed of their interest in recreational topics. Topology had its origin in Euler's analysis of a puzzle about crossing bridges. ... What is mathematics, after all, except the solving of puzzles?"

Martin Gardner

What Is A Manifold?

A manifold is an abstract generalization, perhaps
higher-dimensional, of a curve or surface.

There are perhaps more definitions of "manifold" in the literature
than any other mathematical object.

Smooth manifolds (many flavors)

Topological manifolds

Topological or smooth manifolds with boundary

We will consider only topological manifolds without boundary, and
discuss some exotic examples related to the definition.

Minimal Definition

A *manifold* is a locally Euclidean topological space: Every point has an open neighborhood homeomorphic to $\mathbb{R}^n$ for some n (or to an open ball in $\mathbb{R}^n$).

Such a space need not be connected, but the connected (path) components are clopen and are themselves locally Euclidean. No loss of generality to assume space is connected.

If X is connected and locally Euclidean, the dimension is constant (well defined by Brouwer's Invariance of Domain Theorem).

n-manifold = manifold of constant dimension n.

First Problem: Is It Hausdorff?

A locally Euclidean topological space is always T_1, but not necessarily Hausdorff.

Standard example:

Let X be $\mathbb{R} \cup \{0'\}$, with the topology consisting of all open subsets of $\mathbb{R}$ and all sets consisting of $0'$ along with an open neighborhood or deleted open neighborhood of 0. Since $\mathbb{R}$ is an open subset of X, every point of $\mathbb{R}$ has a locally Euclidean neighborhood; and $0'$ does also, e.g. $\{0'\} \cup (\mathbb{R} \setminus \{0\})$. Thus X is locally Euclidean (and 1-dimensional). But 0 and $0'$ do not have disjoint neighborhoods.

We thus revise our definition:

A manifold is a Hausdorff locally Euclidean topological space (usually connected).

Properties:

    locally compact

    completely regular

    first countable

    (locally) path-connected

    locally contractible

    components have cardinality $2^{\aleph_0}$

Is A Manifold Metrizable?

Theorem:

Let X be a connected Hausdorff locally Euclidean topological space. The following are equivalent:

- X is metrizable.
- X is second countable.
- X is σ-compact.
- X is Lindelöf.
- X is paracompact.
- X is a Polish space.

Many authors require manifolds to satisfy these conditions.

Big Question: Are these conditions automatic?

Well-Ordered Sets (Ordinals)

A *well-ordered set* is a totally ordered set in which every nonempty subset has a smallest element (equivalently, which has no strictly decreasing sequences).

Generic well-ordered set:

Ordinal: order type of a well-ordered set.

The set of all countable ordinals is an uncountable well-ordered set Ω. No countable cofinal subset.

The Long Line

Beginning with Ω, between each pair of consecutive points put a copy of $(0,1)$. The resulting ordered set is called the *long ray* L_0. Any bounded segment of L_0 is an interval; but L_0 is "too long" to be an interval (no countable cofinal set). Give L_0 the order topology. L_0 is normal (hence Hausdorff).

L_0 is not quite locally Euclidean since it has an endpoint 0. Can make two locally Euclidean spaces ("1-manifolds") from it:

Let L_+ be $L_0 \setminus \{0\}$. L_+ is the *half-long line.*

Let L be $L_0 \cup (-L_0)$ with the endpoints identified. L is the *long line.*

These spaces are normal, but not metrizable, separable, or contractible. L is countably compact and sequentially compact.

The only connected 1-manifolds are $\mathbb{R}$, L, L_+, and S^1.

Some Related 2-Manifolds

Consider the following Cartesian products

$L \times L$, the *big plane.*

$L \times L_+$, the *big half-plane.*

$L_+ \times L_+$, the *big quadrant.*

$L \times \mathbb{R}$, the *wide plane* or *long strip.*

$L_+ \times \mathbb{R}$, the *half-long strip.*

$L \times S^1$, the *long cylinder.*

$L_+ \times S^1$, the *half-long cylinder.*

No two of these are homeomorphic.

In the big plane $L \times L$, there are some obvious copies of L: the axes and the diagonals.

Interesting fact: the diagonals are embedded in $L \times L$ in a different way than the axes! A similar phenomenon holds in the other products.

This is used to prove that the spaces are not homeomorphic.

Another example: Form $L_0 \times S^1$, and identify the initial circle to a point. This is the *big cone* or *big disk*. This space is not homeomorphic to any of the previous ones.

The slick way to distinguish these spaces is via their Stone-Čech compactifications. If X is completely regular, its Stone-Čech compactification βX and corona space $\beta X \setminus X$ are usually horrendous huge pathological spaces. But for these spaces they are quite manageable.

The long ray L_0 has the unusual property that its Stone-Čech compactification is the same as its one-point compactification. The Stone-Čech compactification of the long line L is a two-point compactification, one point at $+\infty$ and one at $-\infty$.

The corona space of the big plane is a "big square" consisting of vertical copies of the long line on the left and right, horizontal copies at the top and bottom, meeting at the corners, which are topologically distinct points. The corona space of the big disk is an ordinary circle. The other examples are similar.

Yet another example: the wedge in the big plane (or big quadrant) between the horizontal axis and the diagonal is the *big octant*.

Copies of the octant (with appropriate edges) can be strung together with any orientations to make new examples.

Two copies back-to-back make the big quadrant.

Four copies with alternate orientation make the big half-plane.

The first and last edges can also be identified together. Eight copies with alternate orientation with first and last edge joined make the big plane.

Even transfinitely many (up to Ω) can be strung together. In this way, uncountably many ($2^{\aleph_1}$) mutually nonhomeomorphic examples can be made. All are normal, but not metrizable or separable. None are contractible despite having trivial algebraic topology.

2-Manifolds of a Different Color

We now describe the construction of a large family of nonmetrizable 2-manifolds with much different properties. There are two basic examples, the Prüfer manifold and the Moore manifold.

For the Prüfer manifold (first described by T. RADÓ in 1925), begin with an uncountable collection $\{\mathbb{R}_a^2 : a \in \mathbb{R}\}$ of copies of the plane, and an additional copy $\mathbb{R}_+^2$ of the open upper half plane. Take the separated union U of the $\mathbb{R}_a^2$ and $\mathbb{R}_+^2$; then U is a 2-manifold. Identify the open upper half plane of each $\mathbb{R}_a^2$ with $\mathbb{R}_+^2$ by

$$(x, y) \in \mathbb{R}_a^2 \ (y > 0) \ \longleftrightarrow \ (xy + a, y) \in \mathbb{R}_+^2$$

and let P be the quotient space.

Think of having one copy of the open upper half plane, and a copy of the closed lower half plane for each $a \in \mathbb{R}$. As one approaches the point $(a, 0)$ through the upper half plane, a point of the x-axis in the a'th copy of the lower half plane is approached; which point depends on what direction $(a, 0)$ is approached from.

The Prüfer manifold is Hausdorff and locally Euclidean, but is not separable. It is not normal, because the closed subspace consisting of the upper half plane and all the copies of the x-axis is separable but contains a closed subspace consisting of an uncountable number of pairwise disjoint relatively clopen sets (a cardinality argument then shows that the conclusion of Urysohn's Lemma cannot hold.)

But P is contractible. Thus it is very different from all the examples constructed from the long line.

The Moore Manifold

Instead of adding a closed lower half plane at each $a \in \mathbb{R}$, we can simply add a copy L_a of $[0, \infty)$ to the open upper half plane. Informally, we approach a point of L_a from the upper half plane by approaching $(a, 0)$ along a certain angular direction from the vertical on either side.

The Moore manifold M has similar properties to the Prüfer manifold P: it is Hausdorff, locally Euclidean, connected, and contractible, but not normal. But unlike P, M is separable: points in the open upper half plane with rational coordinates are dense.

The Moore manifold is a quotient of the Prüfer manifold (collapse semicircles centered at 0 in each of the lower half-planes to points). If each ray in the Moore plane is collapsed to a point, the Niemytzki tangent disk space is obtained.

Another Viewpoint

There is another way to look at the construction which may be helpful. If $\mathbb{D}$ is the open unit disk in $\mathbb{R}^2$, and a (relatively) closed ray from the center to the edge is removed, the remainder is homeomorphic to $\mathbb{D}$ (hence to $\mathbb{R}^2$). Similarly, if a (relatively) closed wedge is removed, what is left is homeomorphic to $\mathbb{D}$.

Think of this in reverse: if a is a point on the boundary of $\mathbb{D}$, a ray in from a can be added to the disk. The disk can even be cut along a ray from a and a wedge inserted.

This can be done at finitely many points, or even countably many points (the rays or wedges don't have to go in very far), to still end up with a disk. But, remarkably, the rays or wedges can be added at *every* point, although one does not end up with $\mathbb{D}$.

Adding rays at each point, the resulting space is M.

Adding wedges at each point, the resulting space is P.

Key Observation: We added a lower half-plane or ray at every point of $\mathbb{R}$. But more generally, if A is any subset of $\mathbb{R}$, we can just add the lower half-planes or rays only at points of A to get manifolds P_A and M_A. Actually, if A and B are any disjoint subsets of $\mathbb{R}$, we can add lower half-planes at the points of A and rays at the points of B.

If $A, B \subseteq \mathbb{R}$, when is $P_A \cong P_B$ and/or $M_A \cong M_B$?

Necessary Condition: If $P_A \cong P_B$ or $M_A \cong M_B$, then there are countable subsets C of A and D of B such that $A \setminus C$ is homeomorphic to $B \setminus D$. (Possibly sufficient also)

So how many homeomorphism classes are there?

Interlude

Proposition:

Let $A \subseteq \mathbb{R}$. Then there are exactly $2^{\aleph_0}$ subsets of $\mathbb{R}$ homeomorphic to A.

Proof: (Kuratowski) If ϕ is a homeomorphism of A onto $B \subseteq \mathbb{R}$, regard ϕ as a continuous function from A to $\mathbb{R}$. Since A is separable, any continuous function from A is completely determined by its values on a countable dense set. Thus there are only as many continuous functions from A to $\mathbb{R}$ as there are sequences in $\mathbb{R}$, i.e. $2^{\aleph_0}$.

Corollary:

There are $2^{2^{\aleph_0}}$ homeomorphism classes of subsets of $\mathbb{R}$.

Thus there are $2^{2^{\aleph_0}}$ homeomorphism classes of P_A's and the same number of homeomorphism classes of M_A's. If A is uncountable, then P_A cannot be homeomorphic to any M_B, since the M_B are all separable and P_A is separable if and only if A is countable.

Actually, if A is countable P_A and M_A are both homeomorphic to $\mathbb{R}^2$. If A is uncountable, P_A and M_A are not normal, hence not metrizable.

All the P_A and M_A are contractible. Thus we have $2^{2^{\aleph_0}}$ pairwise nonhomeomorphic contractible (nonmetrizable) 2-manifolds, either separable or nonseparable.

These examples are all nonnormal. I do not know an example of a normal contractible 2-manifold which is not metrizable. (May be related to the *normal Moore space conjecture*)

Returning to the disk picture, adding rays or wedges just at points of a subset A of the circle, one ends up with M_A or P_A.

If A and B are disjoint subsets of the circle, we may add wedges at points of A and rays at the points of B to get a manifold $D_{A,B}$. We have $P_A = D_{A,\emptyset}$ and $M_B = D_{\emptyset,B}$. These are all contractible, but not normal if A or B is uncountable.

We have $D_{A,B}$ homeomorphic to $D_{A',B'}$ only if (possibly if and only if) $A \cup B$ differs by a countable set from a set homeomorphic to $A' \cup B'$ under a homeomorphism sending A to A' and B to B' (modulo countable sets). In particular, if A and B are uncountable, then $D_{A,B}$ is not homeomorphic to any $P_{A'}$ or $M_{B'}$.

The Prüfer or Moore construction can also be done replacing the upper half plane with the big half plane or long or half-long strip (or other of the previous examples), and in the Prüfer case attaching a half-long strip in place of the lower half-plane at some or all the points of the x-axis, leading to a larger family of examples. These will not be contractible, however. But they won't be the same as the earlier examples since they are not normal.

More generally, any two of the above manifolds with an open interval on the edge can be joined there, or a Prüfer or Moore construction can be done along it. These can be repeated in mind-boggling variety.

I imagine these examples barely scratch the surface (pun intended!) of the world of possible connected 2-manifolds with trivial algebraic topology.

So we have a huge supply $(2^{2^{\aleph_0}})$ of distinct contractible (nonmetrizable) 2-manifolds. What about metrizable ones?

Theorem:

Any contractible metrizable 2-manifold is homeomorphic to $\mathbb{R}^2$.

We have an (almost) better result:

Theorem:

Any simply connected open subset of $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

The arguably best proof of this theorem uses the Riemann Mapping Theorem of Complex Analysis, which gives a much stronger conclusion.

A simply connected metrizable 2-manifold is homeomorphic to $\mathbb{R}^2$ or S^2.

What about in $\mathbb{R}^3$, or $\mathbb{R}^n$, $n > 3$?

Let T_0 be a solid (unknotted) torus in $\mathbb{R}^3$, and T_1 a thinner solid torus embedded in T_0 as in the figure:
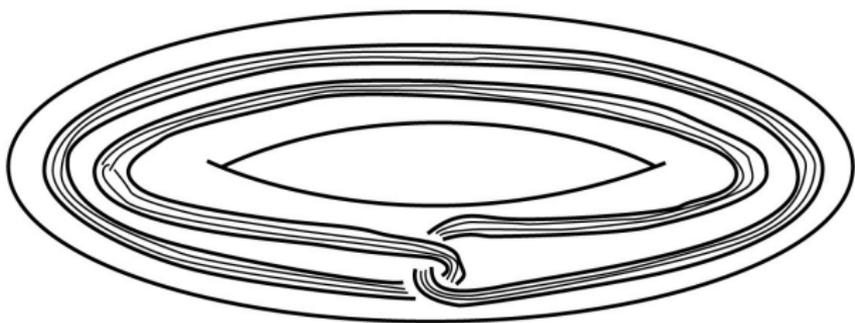
For each k let T_{k+1} be a thin solid torus embedded in T_k in the same way. Note that each T_k is unknotted in $\mathbb{R}^3$.
Let $W = \cap_{k=1}^{\infty} T_k$. W is called the *Whitehead continuum* (the exact space depends on the details of the construction).

The complement of W is more interesting. Regard the construction as being done in the one-point compactification S^3 of $\mathbb{R}^3$, and let $E = S^3 \setminus W$. Then E is a connected, noncompact 3-manifold, and E is contractible. E is called the *Whitehead manifold*. By taking a homeomorphism of S^3 sending a point of W to ∞, E can be regarded as an open subset of $\mathbb{R}^3$.

No neighborhood of ∞ in the one-point compactification $E^\dagger$ of E is simply connected [$E^\dagger$ is homeomorphic to the quotient S^3/W of S^3 obtained by collapsing W to a point.] Thus $E^\dagger$ is not a manifold, so E is not homeomorphic to $\mathbb{R}^3$.

The construction can be jazzed up to give a family of $2^{\aleph_0}$ pairwise nonhomeomorphic contractible open sets in $\mathbb{R}^3$.

A Little Algebraic Topology

What can we say about the fundamental group of a manifold? Probably nothing in general (except a crude cardinality bound $2^{\aleph_0}$).

The fundamental group of a metrizable manifold is countable.

Seifert and Threlfall state that every finitely generated group is the fundamental group of a closed 4-manifold. But their proof clearly only works for finitely *presented* groups. In fact:

> The fundamental group of a compact manifold (even with boundary) is finitely presented.
>
> There are only countably many compact manifolds (even with boundary) up to homeomorphism (J. CHEEGER).

There are only countably many finitely presented groups, but uncountably many finitely generated groups.

Actually Seifert and Threlfall's statement is somewhat ambiguous, and their result could hold for noncompact (metrizable) 4-manifolds.

The following remarkable theorem is known:

Theorem:

Every countable group occurs as the fundamental group of a metrizable manifold. In fact, every countable group occurs as the fundamental group of an open set in $\mathbb{R}^5$.

Is this true also for 4-manifolds, or even open sets in $\mathbb{R}^4$?