Ideal Structure of NF Algebras and The UCT Revisited

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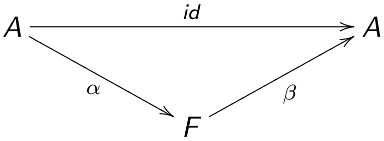
NF Algebras

Part 1. Ideal Structure of NF Algebras

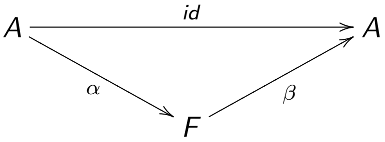
Recall that a C*-algebra A is *nuclear* if, for any $x_1, \ldots, x_n \in A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra F and completely positive contractions $\alpha : A \to F$ and $\beta : F \to A$, such that $\|\beta \circ \alpha(x_i) - x_i\| < \epsilon$ for all i.



This is only one of many characterizations of nuclear C*-algebras.

Definition:

A separable C*-algebra A is an *NF algebra* if, for any $x_1, \ldots, x_n \in A$ and $\epsilon > 0$ there is a finite-dimensional C*-algebra F and completely positive contractions $\alpha : A \to F$ and $\beta : F \to A$ such that $\|\beta \circ \alpha(x_i) - x_i\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all i, j.



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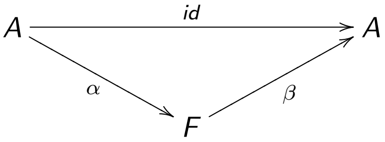

An NF algebra is a separable C*-algebra in which not only the complete order structure but also the multiplication can be approximately modeled in finite-dimensional C*-algebras.

Here are a few of the many characterizations of NF algebras:

Theorem (B.-Kirchberg):

Let A be a separable C*-algebra. The following are equivalent:

(i) A is an NF algebra.

(ii) A is nuclear and quasidiagonal.

(iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C*-algebras in which the connecting maps are completely positive contractions (and asymptotically multiplicative).

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$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \longrightarrow A$$

Such a system is called an *NF system* for A.

If $(A_n, \phi_{n,n+1})$ is an NF system for A, there is a completely positive contraction $\phi_n : A_n \to A$, and $\cup_n \phi_n(A_n)$ is dense in A. But $\phi_n(A_n)$ is not a subalgebra of A in general.

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A quotient of an NF algebra is not necessarily NF. In fact, any separable nuclear C*-algebra is a quotient of an NF algebra [if A is separable and nuclear, then the cone over A is an NF algebra by Voiculescu.]

Ideals in an NF Algebra

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Example. Let $A_n = \mathbb{M}_n$,

$$\phi_{n,n+1}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

The inductive limit A is isomorphic to $\mathbb{K} + \mathbb{C}1$. Each A_n is simple, but A is not simple.

But there is something that can be said about ideals in a generalized inductive limit. Let $A = \lim_{\rightarrow}(A_n, \phi_{n,n+1})$, and let J be an ideal in A. Since $\phi_n(A_n)$ is not a subalgebra of A, $\phi_n^{-1}(J)$ is not a subalgebra of A_n in general.

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However, since ϕ_n is positive, $\phi_n^{-1}(J) \cap A_{n+}$ is a (closed) hereditary cone in A_{n+}, so its span is a hereditary C*-subalgebra J_n of A_n (not an ideal in general.) Since A_n is finite-dimensional, J_n is a corner, i.e. $J_n = p_n A_n p_n$ for a projection $p_n \in A_n$.

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$\phi_{n,n+1}(p_n)$ is not a projection in A_{n+1} in general. However, p_{n+1} is a unit for $\phi_{n,n+1}(p_n)$.

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Definition:

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Definition:

An ideal J in an NF algebra A is an *NF ideal* if A/J is an NF algebra.

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Suppose J is induced from the system $(A_n, \phi_{n,n+1})$, and let p_n be the projection in A_n corresponding to J_n. Define a generalized inductive system as follows: let $B_n = (1 - p_n)A_n(1 - p_n)$, and define $\psi_{n,n+1} : B_n \to B_{n+1}$ by

$$\psi_{n,n+1}(x) = (1 - p_{n+1})\phi_{n,n+1}(x)(1 - p_{n+1})\,.$$

It is routine to check that this is indeed a generalized inductive system, hence an NF system, and that the generalized inductive limit is naturally isomorphic to A/J.

Using a similar (but simplified) argument, one can show:

Proposition:

Let J be an ideal in an NF algebra A. If J has a quasicentral approximate unit of projections, then J is an NF ideal.

This can also be proved using the known result that the quotient of a quasidiagonal C*-algebra by an ideal with a quasicentral approximate unit of projections is quasidiagonal.

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We say an ideal J in a C*-algebra A is *locally approximately split* if, for every $x_1, \ldots, x_n \in A/J$ and $\epsilon > 0$, there is a completely positive contraction $\sigma : A/J \to A$ such that $\|\pi \circ \sigma(x_i) - x_i\| < \epsilon$ and $\|\sigma(x_i x_j) - \sigma(x_i)\sigma(x_j)\| < \epsilon$ for all i, j, where $\pi : A \to A/J$ is the quotient map.

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Proposition:

A locally approximately split ideal in an NF algebra is an NF ideal.

There is a "converse" to the theorem. Suppose J is an NF ideal in an NF algebra A. We want to show that J is an induced ideal from some NF system for A. We outline the argument.

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Suppose we are given $x_1, \ldots, x_n$ in A and $y_1, \ldots, y_m$ in J. Let $\pi : A \to A/J$ be the quotient map, and $\sigma : A/J \to A$ a completely positive contractive cross section for π.

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Since A/J is an NF algebra, there are $\bar{\alpha}_1 : A/J \to F_1$ and $\bar{\beta}_1 : F_1 \to A/J$ (F_1 finite-dimensional) such that $\bar{\beta}_1 \circ \bar{\alpha}_1(\pi(x_i)) \approx \pi(x_i)$ (within ϵ) for all i and $\bar{\alpha}_1$ is approximately multiplicative on the $\pi(x_i)$.

If $\alpha_1 = \bar{\alpha}_1 \circ \pi$ and $\beta_1 = \sigma \circ \bar{\beta}_1$, then $\beta_1 \circ \alpha_1(x_i) - x_i$ and $\beta_1 \circ \alpha_1(x_i x_j) - \beta_1(\alpha_1(x_i)\alpha_1(x_j))$ are approximately in J for all i, j.

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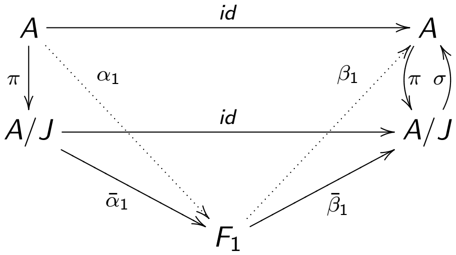

Using a quasicentral approximate unit for J, choose $h \in J_+$, $\|h\| \leq 1$, such that h almost commutes with the x_i, $\beta_1 \circ \alpha_1(x_i)$, $\beta_1 \circ \alpha_1(x_i x_j)$, and $\beta_1(\alpha_1(x_i)\alpha_1(x_j))$, and such that h is approximately a unit for the y_i, $\beta_1 \circ \alpha_1(x_i) - x_i$, and $\beta_1 \circ \alpha_1(x_i x_j) - \beta_1(\alpha_1(x_i)\alpha_1(x_j))$.

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Let $\alpha_2 : A \to F_2$ and $\beta_2 : F_2 \to A$ (F_2 finite-dimensional) be completely positive contractions such that $\beta_2 \circ \alpha_2$ is approximately the identity and α_2 is approximately multiplicative on all elements defined so far.

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Let $\alpha = \alpha_1 \oplus \alpha_2 : A \to F_1 \oplus F_2$ and $\beta : F_1 \oplus F_2 \to A$, where

$$\beta(x, y) = (1 - h)^{1/2}\beta_1(x)(1 - h)^{1/2} + h^{1/2}\beta_2(y)h^{1/2}$$

To get an NF system, set $A_1 = F_1 \oplus F_2$. At the next stage, reduce ϵ and expand $\{x_i\}$ by throwing in all the images of matrix units of $F_1 \oplus F_2$ and more elements of a dense subset of A, and expand $\{y_i\}$ by throwing in the images of the matrix units of F_2 (which lie in J) as well as more elements of a dense subset of J.

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The ideal J is induced by this NF system since $\beta^{-1}(J)$ contains close approximants to $y_1, \ldots, y_m$.

So the conclusion is:

Theorem:

Let A be an NF algebra, J an ideal of A. Then J is an NF ideal if and only if J is induced by some NF system for A.

It would be nice to get a single NF system for an NF algebra A such that all NF ideals can be induced from this system. It is really only necessary to be able to do the previous construction with two NF ideals J and K (or finitely many) simultaneously.

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If $J \cap K$ and $J + K$ are also NF ideals, it appears the construction can be made to work with some technical complications. It is true that $J \cap K$ is always an NF ideal, since $A/(J \cap K)$ can be embedded in $A/J \oplus A/K$ and hence is an NF algebra. But it is not obvious that $J + K$ is always NF.

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The construction at least appears to work for residually NF algebras (strongly quasidiagonal nuclear C*-algebras).

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Is every stably finite separable nuclear C*-algebra an NF algebra?

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The theorem gives a potential approach to showing that a separable nuclear C*-algebra B with a faithful tracial state τ must be an NF algebra. Any separable nuclear C*-algebra is a quotient of an NF algebra, so B is a quotient of an NF algebra A, and τ may be regarded as a tracial state on A. If J is the kernel of τ, i.e.

$$J = \{x \in A : \tau(x^*x) = 0\}$$

then J is the kernel of the quotient map from A to B. So it suffices to show that the kernel of a tracial state on an NF algebra is an induced ideal.

The Universal Coefficient Theorem

Part 2. The UCT Revisited
(or, Does KK-Theory See Complex Structure?)

The general validity of the Universal Coefficient Theorem for separable nuclear C*-algebras is the "elephant in the closet" in the subject of Classification. Many results include validity of the UCT as a hypothesis.

We will break down this question into several subquestions and explore a possible obstruction.

The Universal Coefficient Theorem

Statement: For many pairs (A, B) of separable C*-algebras, there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK^*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

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The class $\mathcal{N}$ of all separable nuclear C*-algebras A for which this sequence is valid for all separable C*-algebras B is called the *Bootstrap Class*.

Theorem (Rosenberg-Schochet):

The class $\mathcal{N}$ is precisely the smallest class of separable nuclear C*-algebras containing $\mathbb{C}$ and closed under several standard bootstrap operations. $\mathcal{N}$ is also precisely the class of separable nuclear C*-algebras which are KK-equivalent to commutative C*-algebras.

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Main Question:

Is $\mathcal{N}$ the class of all separable nuclear C*-algebras?

Recall where the maps γ and δ come from:

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For γ:
If $\mathbf{x} \in KK(A, B)$, then using the pairing

$$KK(\mathbb{C}, A) \times KK(A, B) \to KK(\mathbb{C}, B)$$

and the isomorphism $K_0(D) \cong KK(\mathbb{C}, D)$, right Kasparov product with $\mathbf{x}$ gives a homomorphism from $K_0(A)$ to $K_0(B)$.

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Similarly, a map from $K_1(A) \cong KK(S\mathbb{C}, A)$ to $K_1(B) \cong KK(S\mathbb{C}, B)$ is obtained.

The δ is more indirect. There is a natural identification of $KK(A,B)$ with $Ext(A,SB)^{-1}$, the group of invertible extensions of A by $SB\otimes\mathbb{K}$. Any such extension E defines a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(SB) & \longrightarrow & K_0(E) & \longrightarrow & K_0(A) \\ \partial\uparrow & & & & \downarrow\partial \\ K_1(A) & \longleftarrow & K_1(E) & \longleftarrow & K_1(SB) \end{array}$$

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The connecting maps ∂ are exactly the maps induced by γ. If $\mathbf{x} \in ker(\gamma) \subseteq KK(A, B)$, the connecting maps are 0, so the cyclic exact sequence gives two short exact sequences

$$0 \to K_0(SB) \cong K_1(B) \to K_0(E) \to K_0(A) \to 0$$

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Thus there is a degree one map κ from $ker(\gamma)$ to $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_*(B))$. If A is in the bootstrap class, κ is a bijection and δ is its inverse.

So if A and B are separable C*-algebras, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ker(\gamma) & \longrightarrow & KK^*(A,B) & \xrightarrow{\gamma} & image(\gamma) & \longrightarrow & 0 \\ & & \kappa\downarrow & & \downarrow id & & \downarrow\iota & & \\ & & \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A),K_*(B)) & & KK^*(A,B) & \xrightarrow{\gamma} & \mathrm{Hom}(K_*(A),K_*(B)) & & \end{array}$$

with the first row exact. The map ι is injective (by definition), but κ is not obviously either injective or surjective.

The UCT holds for a pair (A, B) if and only if all three of the following are true:

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Failure of surjectivity of either ι or κ means $KK^*(A, B)$ is "smaller than the UCT would predict."

Failure of injectivity of κ means $KK^*(A, B)$ is "larger than the UCT would predict."

So we can reduce the UCT to three questions. Let A and B be separable C*-algebras, with A nuclear. (We usually ask these questions for a fixed A, letting B vary.)

Question 1:

Is ι always surjective, i.e. is γ always surjective?

Question 2:

Is κ always injective?

Question 3:

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Is κ always surjective?

These questions are not independent: for example, for a given A, if (1) and (2) have positive answers for all B with $K_*(B)$ divisible, then all three have positive answers for all B (Rosenberg-Schochet).

Opposite and Conjugate C*-Algebras

If A is a C*-algebra, then its *opposite algebra* is the C*-algebra A^{op} is A with the same linear structure, adjoint, and norm, and with multiplication

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Question:

Does there exist a separable [nuclear] C*-algebra A such that A and A^{op} are not KK-equivalent?

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$K_*(A)$ and $K_*(A^{op})$ are naturally isomorphic since the projections, unitaries, and partial isometries in A^{op} are just the projections, unitaries, and partial isometries in A respectively. Thus if ι is surjective, there is an $\mathbf{x} \in KK(A, A^{op})$ with $\gamma(\mathbf{x})$ an isomorphism.

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Proposition (Rosenberg-Schochet):

Let A and B be separable C*-algebras with $A \in \mathcal{N}$. Let $\mathbf{x} \in KK(A, B)$. If $\gamma(\mathbf{x}) \in \mathrm{Hom}(K_*(A), K_*(B))$ is an isomorphism, then $\mathbf{x}$ is a KK-equivalence.

So if $A \in \mathcal{N}$, then A and A^{op} are KK-equivalent.

If A is a separable nuclear C*-algebra which is not KK-equivalent to A^{op}, the Proposition shows that at least one of Questions (1) – (3) has a negative answer for some pair (A, B). Which pair(s) and which question(s)?

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The first guess might be that Question (1) has a negative answer (i.e. γ is not surjective) for (A, A^{op}), although this is not clear. In fact, if there is a separable nuclear A with trivial K-theory which is not KK-equivalent to A^{op}, then it must be that $KK(A, A)$ is not 0, i.e. κ is not injective for the pair (A, A).

The proof of the Rosenberg-Schochet Proposition also shows:

Proposition:

Let A and B be separable C*-algebras such that the UCT sequence holds for (A, A) and (A, B). Let $\mathbf{x} \in KK(B, A)$. If $\gamma(\mathbf{x}) \in \mathrm{Hom}(K_*(B), K_*(A))$ is an isomorphism, then $\mathbf{x}$ is a KK-equivalence.

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Applying this to $B = A^{op}$, we obtain that if A is not KK-equivalent to A^{op}, then at least one of the three questions has a negative answer for either (A, A) or (A, A^{op}).

Construction of a counterexample may be difficult. An A not isomorphic to A^{op} can be constructed as a stable homogeneous C*-algebra over a suitable lens space with nontrivial Dixmier-Douady invariant. But, like all Type I C*-algebras, this A is KK-equivalent to A^{op}; there is a homotopy result about lens spaces which "explains" this.

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No counterexample can be constructed simply with groups or group actions, or more generally with C*-algebras which are the complexifications of real C*-algebras. But perhaps a more sophisticated construction involving cocycles could yield an example.

Instead of using A^{op}, I think it is conceptually better to consider the *conjugate algebra* of the C*-algebra A. This is the C*-algebra A^c whose underlying set, *-ring structure, and norm are identical to A, but where the scalar multiplication is conjugated.

A^c is isomorphic to A^{op} as a (complex) C*-algebra via $x \mapsto x^*$, hence is anti-isomorphic to A as a (complex) C*-algebra. The identity map from A to A^c is a *-isomorphism of *real* C*-algebras.

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The previous question can be rephrased:

Question:

Does there exist a separable [nuclear] C*-algebra A such that A and A^c are not KK-equivalent?

The only difference between A and A^c is the scalar multiplication. Scalar multiplication does not enter into the computation of the K-groups, i.e. K-theory does not "see" the difference between A and A^c.

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The question whether $KK(A, A^c)$ is always "the same as" $KK(A, A)$ (i.e. whether A and A^c are KK-equivalent) is really asking whether KK-theory "sees" the complex structure of A.

It is obvious that $A \mapsto A^c$ (or $A \mapsto A^{op}$) is functorial, i.e. a (complex-linear) *-homomorphism from A to B gives a (complex-linear) *-homomorphism from A^c to B^c. More is true:

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If $\mathcal{E}$ is a Hilbert B-module, then $\mathcal{E}$ with the same addition, module structure, and inner product, but with scalar multiplication conjugated, is naturally a Hilbert B^c-module we denote $\mathcal{E}^c$. A $T \in \mathcal{L}(\mathcal{E})$ gives an adjointable operator on $\mathcal{E}^c$, and the "identity map" gives an identification of $[\mathcal{L}(\mathcal{E})]^c$ with $\mathcal{L}(\mathcal{E}^c)$.

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We thus get a natural identification of $KK(A, B)$ with $KK(A^c, B^c)$ for any A and B, which is natural even at the level of Kasparov modules. In particular, for any A and B, $KK(A^c, B)$ is naturally isomorphic to $KK(A, B^c)$.

Thus $A \mapsto A^c$ gives an involution on the category of separable C*-algebras and *-homomorphisms, and also on the category **KN** whose objects are separable nuclear C*-algebras and for which the morphisms from A to B are the elements of $KK(A, B)$, with composition via the Kasparov product.

On the full subcategory **KKN** with objects in $\mathcal{N}$, this involution is "trivial" (preserves isomorphism classes).

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Question:

Is this involution trivial on **KN**?

Recall the definition of a (pre-)Hilbert B-module:

Definition

A *pre-Hilbert B-module* is a right B-module $\mathcal{E}$ which is compatibly a complex vector space, equipped with a *B-valued pre-inner product* $\langle \cdot\,,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to B$ with the following properties for $\xi, \eta, \zeta \in \mathcal{E}$, $b \in B$, $\lambda \in \mathbb{C}$:

(i) $\langle \xi, \eta + \zeta \rangle = \langle \xi, \eta \rangle + \langle \xi, \zeta \rangle$ and $\langle \xi, \lambda\eta \rangle = \lambda\langle \xi, \eta \rangle$

(ii) $\langle \xi, \eta b \rangle = \langle \xi, \eta \rangle b$

(iii) $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$

(iv) $\langle \xi, \xi \rangle \geq 0$ (as an element of B).

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(iii) $\langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$

(iv) $\langle \xi, \xi \rangle \geq 0$ (as an element of B).

The compatible complex-linearity is perhaps underappreciated. If this is weakened to real-linearity, the larger class of real (pre-)Hilbert B-modules is obtained, and KK-theory using this larger class of Hilbert modules is Kasparov's real KK-theory $KK_{\mathbb{R}}$.

Proposition:

Let A and B be separable (complex) C*-algebras. Then

$$KK_{\mathbb{R}}(A, B) \cong KK(A, B) \oplus KK(A^c, B)$$

In particular, if A is a separable complex C*-algebra, $KK_{\mathbb{R}}(A, A)$ is isomorphic to $KK(A, A) \oplus KK(A^c, A)$. $KK_{\mathbb{R}}(A, A)$ is a $\mathbb{Z}_2$-graded ring with $KK(A, A)$ the degree 0 part. (This is a different grading than the grading of $KK^*_{\mathbb{R}}(A, A)$ by degrees, which is a $\mathbb{Z}_8$-grading in the real case, although it collapses to a $\mathbb{Z}_2$-grading if A is complex).

If the UCT fails because there is a separable nuclear A which is not KK-equivalent to A^c, it is conceivable that there could be a substitute UCT involving real KK-theory, taking into account the nontriviality of the involution on $\mathbf{KN}$.

If the UCT fails because there is a separable nuclear A which is not KK-equivalent to A^c, it is conceivable that there could be a substitute UCT involving real KK-theory, taking into account the nontriviality of the involution on **KN**.

There is a UCT for real C*-algebras due to J. Boersema. However, this sequence gives no additional information for complex C*-algebras: if A and B are separable complex C*-algebras, Boersema's exact sequence applies only if $A \in \mathcal{N}$, in which case it essentially gives two copies of the Rosenberg-Schochet UCT sequence.

One last observation. If A is a separable C*-algebra, then $KK(A, A)$ is a ring. The set of finite sums of elements of $KK(A, A)$ which factor through A^c, i.e. are of the form $\mathbf{xy}$ for some $\mathbf{x} \in KK(A, A^c)$ and $\mathbf{y} \in KK(A^c, A)$, is an ideal in $KK(A, A)$, which is all of $KK(A, A)$ if A and A^c are KK-equivalent.

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What information does the quotient ring give?