Nonstable K-Theory for Free Products

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Standing convention:

All C*-algebras, C*-subalgebras, homomorphisms, and free products will be *unital*.

Exceptions: hereditary C*-subalgebras (including ideals), stable algebras

Nonstable K-Theory

Nonstable K_0: If A is a C*-algebra and p, q projections in A with $[p] = [q]$ in $K_0(A)$, is $p \sim q$? Is $p \sim_u q$? Is $p \sim_h q$?

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Nonstable K_1: If $u \in \mathcal{U}(A)$ and $[u] = 0$ in $K_1(A)$, is $u \in \mathcal{U}(A)_{\mathrm{o}}$?

The Unitary Path Group

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$A \mapsto UP(A)$ is functorial, and there is a natural homomorphism γ from $UP(A)$ to $K_1(A)$ which is neither injective nor surjective in general.

A is K_1*-injective* [resp. K_1*-surjective*] if γ is injective [resp. surjective].

Properly Infinite C*-Algebras

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A projection in a (unital) C*-algebra is *very full* if it contains a subprojection equivalent to 1.

p is *splitting* if both p and $1 - p$ are very full.

A contains a splitting projection if and only if A is properly infinite.

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Very full projections in the same K_0-class are not necessarily unitarily equivalent (e.g. 1 and the range projection of an isometry). However:

Corollary:

Splitting projections in the same K_0-class are unitarily equivalent.

The Main Questions

Question 1.

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It is easy to see that a properly infinite C*-algebra is K_1-surjective: if p is a splitting projection in A, then the (nonunital) embedding of pAp into A extends to an embedding of the (nonunital) C*-algebra $pAp \otimes \mathbb{K}$ into A.

Proposition:

If u is a unitary in A with $[u] = 0$ in $K_1(A)$ (i.e. $\gamma(u) = 0$), and if u commutes with a splitting projection, then $u \in \mathcal{U}(A)_o$.

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From this, it follows easily that a purely infinite (simple unital) C*-algebra is K_1-injective, since by functional calculus every unitary is homotopic to a unitary which commutes with a splitting projection.

The condition can be relaxed to approximate commutativity:

Proposition (Kirchberg, Blanchard–Rohde–Rørdam):

If u is a unitary in A with $[u] = 0$ in $K_1(A)$ (i.e. $\gamma(u) = 0$), and if there is a splitting projection p in A with $\|up - pu\| < 1$, then $u \in \mathcal{U}(A)_o$.

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In particular, if A has a central sequence of splitting projections, then A is K_1-injective.
O_∞ has such a central sequence (O_∞ is isomorphic to an infinite tensor product of copies of itself), as does $A \otimes O_\infty$ for any A, so $A \otimes O_\infty$ is K_1-injective for any (unital) A.

Equivalence of Questions 1 and 2

Theorem (Blanchard-Rohde-Rørdam):

Let A be a properly infinite (unital) C*-algebra. The following are equivalent:

(1) Whenever p and q are splitting projections in A with $[p] = [q]$ in $K_0(A)$, then p and q are homotopic in A.

(1′) Whenever p and q are splitting projections in A with $[p] = [q] = [1]$ in $K_0(A)$, then p and q are homotopic in A.

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(2) $\Rightarrow$ (1): If p and q are splitting projections in a properly infinite C*-algebra A, with the same K_0-class, they are unitarily equivalent via a unitary with trivial K_1-class (correct using a unitary in pAp.)

So if A is properly infinite and K_1-injective, and p and q are splitting projections with the same K_0-class, then p and q are unitarily equivalent via a unitary in $\mathcal{U}(A)_{\text{o}}$, hence homotopic.

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(1′) $\Rightarrow$ (2): If u is a unitary in A with trivial K_1-class, let p be a splitting projection equivalent to 1 and set $q = u^* pu$. If p and q are homotopic, then u is homotopic to a unitary commuting with p, hence is in $\mathcal{U}(A)_{\mathrm{o}}$.

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Thus Question 1 and Question 2 are equivalent.

In Question 1, we may also assume that $p \sim q \sim 1$.

There are, roughly speaking, three possibilities for the outcomes of these Questions (and ones to be discussed later too):

1. The answers are negative in the sense that any possible pathology occurs.
2. Some pathology is ruled out by simple general arguments we have not yet found.
3. Some pathology is nonexistent for subtle and deep reasons.

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3. Some pathology is nonexistent for subtle and deep reasons.

Outcome (3) seems the most unlikely, but would be the most interesting one if it happens.

In the spirit of M. Gromov's famous principle that there is no nontrivial statement which is true for all groups, my feeling is that outcome (1) is the most likely for these Questions.

Unital Free Products

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Free products behave quite differently from tensor products in many ways, including nonstable K-theory.

Example:

$O_2 *_{\mathbb{C}} O_2 \cong O_2 *_{\mathbb{C}} C(\mathbb{T})$. ($O_2$ can be replaced by O_n for any $n < \infty$, but not by O_∞.)

To define a specific isomorphism, let $\{s_1, s_2, t_1, t_2\}$ be the standard generators of $O_2 *_{\mathbb{C}} O_2$ and $\{s_1, s_2, u\}$ the standard generators of $O_2 *_{\mathbb{C}} C(\mathbb{T})$. Define

$$\phi : O_2 *_{\mathbb{C}} O_2 \to O_2 *_{\mathbb{C}} C(\mathbb{T})$$

$$\phi(s_i) = s_i, \phi(t_i) = us_i$$

$$\psi : O_2 *_{\mathbb{C}} C(\mathbb{T}) \to O_2 *_{\mathbb{C}} O_2$$

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Since $C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T}) \cong C^*(\mathbb{F}_2)$, we get

$$O_2 *_{\mathbb{C}} O_2 *_{\mathbb{C}} O_2 \cong O_2 *_{\mathbb{C}} C^*(\mathbb{F}_2)$$

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Let p_1 and q_1 be range projections of some generators in the two copies of O_∞. If p_1 and q_1 are homotopic in $O_\infty *_{\mathbb{C}} O_\infty$, and A, p, q are as in Question 1, using $(1')$ we can reduce to the case where p and q are equivalent to 1_A, i.e. range projections of isometries. Since $1-p$ and $1-q$ are very full, there is then a (unital) homomorphism ϕ from $O_\infty *_{\mathbb{C}} O_\infty$ to A with $\phi(p_1) = p$ and $\phi(q_1) = q$, and the homotopy of p_1 and q_1 gives a homotopy of p and q.

Interesting alternate point of view:

Example (Blanchard-Rohde-Rørdam):

Let A and B be the two natural copies of O_∞ in $O_\infty *_{\mathbb{C}} O_\infty$, and let D be the set of continuous functions from $[0,1]$ to $O_\infty *_{\mathbb{C}} O_\infty$ such that $f(0) \in A$, $f(1) \in B$. D is a continuous field of properly infinite C*-algebras over $[0,1]$.

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Is D properly infinite? Does D contain any nontrivial projections? Is D K_1-injective?

K-Theory of Unital Free Products

The exact sequence of E. Germain and K. Thomsen for amalgamated free products gives the following exact sequence for unital free products of C*-algebras:

$$\begin{array}{ccccc} K_0(\mathbb{C}) \cong \mathbb{Z} & \longrightarrow & K_0(A \oplus B) & \longrightarrow & K_0(A *_{\mathbb{C}} B) \\ \uparrow & & & & \downarrow \\ K_1(A *_{\mathbb{C}} B) & \longleftarrow & K_1(A \oplus B) & \longleftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

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Thus the map $K_1(A \oplus B) \to K_1(A *_{\mathbb{C}} B)$ is injective. This map is induced by the natural inclusions from A and B into $A *_{\mathbb{C}} B$.

In particular:

Corollary.

If A and B are (unital) C*-algebras, then the inclusion from A into $A *_{\mathbb{C}} B$ induces an injective map from $K_1(A)$ to $K_1(A *_{\mathbb{C}} B)$.

The corresponding statement for K_0 is false in general.

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Incidentally, the exact sequence shows that $K_1(O_2 *_{\mathbb{C}} O_2)$ is nontrivial! ($\cong \mathbb{Z}$)
The isomorphism $O_2 *_{\mathbb{C}} O_2 \cong O_2 *_{\mathbb{C}} C(\mathbb{T})$ "explains" this.

What about a nonstable version:

Question 3.

If A and B are (unital) C*-algebras, does the inclusion of A into $A *_{\mathbb{C}} B$ induce an injective map from $UP(A)$ to $UP(A *_{\mathbb{C}} B)$?

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A bolder version would ask whether the map from the free product $UP(A) * UP(B)$ to $UP(A *_{\mathbb{C}} B)$ is always injective.

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A bolder version would ask whether the map from the free product $UP(A) * UP(B)$ to $UP(A *_{\mathbb{C}} B)$ is always injective.

If this is true, then
$UP(C^*(\mathbb{F}_2)) = UP(C(\mathbb{T}) *_{\mathbb{C}} C(\mathbb{T})) = \mathbb{Z} * \mathbb{Z} = \mathbb{F}_2$.

Free Group C*-Algebras

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Let u and v be the canonical unitary generators of $C^*(\mathbb{F}_2)$, and $w = uvu^{-1}v^{-1}$.

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The class of w in $UP(C^*(\mathbb{F}_2))$ is nontrivial (i.e. $UP(C^*(\mathbb{F}_2))$ is nonabelian.)

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Question 4.

Is the class of w in $UP(C^*(\mathbb{F}_2) *_{\mathbb{C}} B)$ nontrivial for every B? (Is $UP(C^*(\mathbb{F}_2) *_{\mathbb{C}} B)$ nonabelian for every B?)

True for at least many B.

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If Question 4 has a positive answer, then Questions 1 and 2 have a negative answer:

$C^*(\mathbb{F}_2) *_{\mathbb{C}} O_\infty$ would be a properly infinite C*-algebra with nonabelian unitary path group, hence not K_1-injective.

Proof of Theorem

A topological group G is *homotopy abelian* if the maps

$$f(x, y) = xy \text{ and } g(x, y) = yx$$

are homotopic as maps from $G \times G$ to G.

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Theorem (Araki, James, Thomas, 1960):

A compact connected Lie group which is homotopy abelian is actually abelian.

Apply this theorem to $U(n)$. Let

$$A = C(U(n) \times U(n), \mathbb{M}_n)$$

Then $\mathcal{U}(A)$ can be identified with the set of continuous functions from $U(n) \times U(n)$ to $U(n)$, so the functions f and g can be regarded as unitaries in A, as can $u(x,y) = x$ and $v(x,y) = y$; $f = uv$ and $g = vu$; $uvu^*v^* = fg^*$ is not in $\mathcal{U}(A)_\text{o}$ by the theorem.

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The unitaries u and v define a homomorphism ϕ from $C^*(\mathbb{F}_2)$ to A; $\phi(w) = fg^*$. Since $\phi(w) \notin \mathcal{U}(A)_\mathrm{o}$, $w \notin \mathcal{U}(C^*(\mathbb{F}_2))_\mathrm{o}$.

The proof shows more: there is a (unital) homomorphism from $\mathbb{M}_n$ to A as constant functions. Thus there is a homomorphism ψ from $C^*(\mathbb{F}_2) *_{\mathbb{C}} \mathbb{M}_n$ to A, and as above $\psi(w) \notin \mathcal{U}(A)_{\text{o}}$, so $w \notin \mathcal{U}(C^*(\mathbb{F}_2) *_{\mathbb{C}} \mathbb{M}_n)_{\text{o}}$ for any n. Thus Question 4 is true for $B = \mathbb{M}_n$.

This is in stark contrast to tensoring with $\mathbb{M}_n$, which can collapse the unitary path group.

Let $\mathcal{B}$ be the class of all C*-algebras B for which Question 4 is true. $\mathcal{B}$ has the following properties:

(i) $\mathbb{M}_n \in \mathcal{B}$.
(ii) $\mathcal{B}$ is closed under direct sums and inductive limits (with injective connecting maps). So every AF algebra is in $\mathcal{B}$.
(iii) If $B \in \mathcal{B}$, then any (unital) subalgebra of B is also in $\mathcal{B}$.
(iv) If some quotient of A is in $\mathcal{B}$, then $A \in \mathcal{B}$. In particular, $\mathcal{T} \in \mathcal{B}$.

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Question 5.

Is there a properly infinite C*-algebra in $\mathcal{B}$? Equivalently, is $O_\infty \in \mathcal{B}$?

A positive answer to Question 5 gives a negative answer to Questions 1 and 2.

By the same argument as with $U(n)$, if B is a C*-algebra whose unitary group is not homotopy abelian, there is a homomorphism ψ from $C^*(\mathbb{F}_2) *_{\mathbb{C}} B$ to the C*-algebra A of bounded continuous functions from $\mathcal{U}(B) \times \mathcal{U}(B)$ to B with $\psi(w) \notin \mathcal{U}(A)_{\rm o}$, so $B \in \mathcal{B}$.

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This suggests a possible approach to Question 5:

Question 6.

Is there a properly infinite C*-algebra whose unitary group is not homotopy abelian?

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Question 6.

Is there a properly infinite C*-algebra whose unitary group is not homotopy abelian?

A positive answer to Question 6 implies that $O_\infty \in \mathcal{B}$, and thus a positive answer to Question 5 and negative answers to Questions 1 and 2.

It could be that the unitary group of any properly infinite C*-algebra is homotopy abelian. But note that the condition that $\mathcal{U}(B)$ not be homotopy abelian is far from necessary for B to be in $\mathcal{B}$ (the unitary group of $\mathbb{C}$ is homotopy abelian!) It is sufficient, for example, that B be embeddable in a C*-algebra whose unitary group is not homotopy abelian.

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A result of James and Thomas seems to suggest that a group like $\mathcal{U}(O_\infty)$ is not homotopy abelian:

Theorem:

Let G be a path-connected topological group. If G is a countable CW-complex with finitely generated integral homology, and G is homotopy abelian, then G is homotopy equivalent to a torus (product of circles). In particular, if G is simply connected, it is contractible.

$\mathcal{U}(O_\infty)$ is not homotopy equivalent to a torus. Perhaps a better candidate would be $\mathcal{U}(P_\infty)$, where P_∞ is the Kirchberg algebra with $K_0(P_\infty) = 0$, $K_1(P_\infty) = \mathbb{Z}$. Then $\mathcal{U}(P_\infty)_{\mathrm{o}}$ is simply connected but not contractible.

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But $\mathcal{U}(O_\infty)$ or $\mathcal{U}(P_\infty)$ does not satisfy the hypotheses of the theorem, since its homology is not finitely generated. (The unitary group of a separable C*-algebra is homotopy equivalent to a countable CW-complex.)

Unitary groups of C*-algebras are not the kind of topological groups topologists normally like to think about!

To see this, use the following result of S. Zhang:

Theorem.

If A is a purely infinite (simple unital) C*-algebra, then, for all $n \geq 0$,

$$\pi_n(\mathcal{U}(A)_{\text{o}}) \cong K_{n+1}(A)$$

This result is reasonable since, for example, $\pi_1(\mathcal{U}(A))$ is the set of homotopy classes of loops of unitaries in A, which is the unitary path group of $(SA)^+$. This is roughly $K_1(SA) \cong K_2(A) = K_0(A)$. But pure infiniteness is needed to destabilize.

Thus, $\pi_1(\mathcal{U}(P_\infty)) = 0$ and $\pi_2(\mathcal{U}(P_\infty)) \cong \mathbb{Z}$.

However, N. C. Phillips has obtained the following result:

Theorem:

The unitary group of any purely infinite C*-algebra is homotopy abelian.

An analysis of Zhang's proof shows that these are infinite loop spaces. The result also applies to the tensor product of a commutative C*-algebra and a purely infinite C*-algebra.

The finite generation hypothesis in the result of James and Thomas thus cannot be removed.

However, N. C. Phillips has obtained the following result:

Theorem:

The unitary group of any purely infinite C*-algebra is homotopy abelian.

An analysis of Zhang's proof shows that these are infinite loop spaces. The result also applies to the tensor product of a commutative C*-algebra and a purely infinite C*-algebra.

The finite generation hypothesis in the result of James and Thomas thus cannot be removed.

So the Question 6 approach to obtaining negative solutions to Questions 1 and 2 does not look promising. But it still could very well be true that some nonsimple properly infinite C*-algebras such as $C^*(\mathbb{F}_2) *_{\mathbb{C}} O_\infty$ or $O_\infty *_{\mathbb{C}} O_\infty$ have unitary groups which are not homotopy abelian, or nonabelian unitary path groups.

For another approach to Question 5, consider examples like the following:

Example:

Let $A = \bigotimes_{\mathbb{F}_2} O_\infty \cong O_\infty$. Let $\mathbb{F}_2$ act on A by permuting the tensor product factors by translating the indices. The full crossed product $B = A \rtimes_\sigma \mathbb{F}_2$ is properly infinite, a quotient of $C^*(\mathbb{F}_2) *_{\mathbb{C}} O_\infty$.

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Is the image of w in $\mathcal{U}(B)_{\rm o}$? If not, the answer to Question 5 is yes.

One could also let A be the infinite unital free product of copies of O_∞.

Unitary Path Groups of Group C*-Algebras

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It seems likely that σ is injective if G is a free group. If not, there is a universal relation holding in $UP(A)$ for every C*-algebra A.

If $s(u, v)$ is a (reduced) word in $\mathbb{F}_2$, then for any topological group H there is an induced continuous map

$$f : H \times H \to H \quad f(x, y) = s(x, y)$$

and one can ask whether f is homotopic to the constant function e_H. The set of words homotopic to the constant function form a normal subgroup of $\mathbb{F}_2$ we will call the *homotopy kernel* of H.

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The kernel of σ for $\mathbb{F}_2$ is exactly the homotopy kernel of $\mathcal{U}(C^*(\mathbb{F}_2))$, and by universality is the intersection of the homotopy kernels for $\mathcal{U}(A)$ for all (unital) C*-algebras A.

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The Araki-James-Thomas result says that if H is a nonabelian compact connected Lie group (e.g. $U(n)$ for $n \geq 2$), then $s(u, v) = uvu^{-1}v^{-1}$ is not in the homotopy kernel of H.

It follows from known results in topology that the homotopy kernel of $U(n)$ is nontrivial for each n (in fact, the quotient of $\mathbb{F}_2$ by the homotopy kernel is nilpotent).

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Question 7.

Is the intersection of the homotopy kernels of $U(n)$ for all n trivial?

This seems reasonable since the intersection of the lower central series in a free group is trivial (Magnus 1935).

A positive answer would show that the homotopy kernel of $\mathcal{U}(C^*(\mathbb{F}_2))$ is trivial, and hence that $\sigma : \mathbb{F}_2 \to UP(C^*(\mathbb{F}_2))$ is injective.

The techniques of Araki-James-Thomas do not seem to yield much information on this problem, but there is some evidence for a positive answer from other topological results.

Question A:

If G is a (reasonable) torsion-free discrete group, is $\sigma : G \to UP(C^*(G))$ injective? an isomorphism (i.e. also surjective)?

The answer to Question A is no if G is not torsion-free: any torsion element of G is in the kernel of σ.

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If the full C*-algebra is replaced by the reduced C*-algebra, the answer is no for G a free group; in fact (Dykema-Haagerup-Rørdam):

$$UP(C_r^*(\mathbb{F}_2)) = K_1(C_r^*(\mathbb{F}_2)) \cong \mathbb{Z}^2 = \mathbb{F}_2/[\mathbb{F}_2, \mathbb{F}_2]$$

Question B:

If G is a (reasonable) torsion-free discrete group, does $\sigma : G \to UP(C_r^*(G))$ induce an injective map [isomorphism] from $G/[G,G]$ to $UP(C_r^*(G))$?

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These questions are a sort of nonstable version of a special case of the Baum-Connes Conjecture: if G is a (reasonable) torsion-free discrete group, then $K_1(C_r^*(G))$ should be the odd homology of $G/[G,G]$, and in particular the natural map from $G/[G,G] = H_1(G/[G,G])$ to $K_1(C_r^*(G))$ should be injective. (It will not be surjective in general, e.g. if $G = \mathbb{Z}^3$.)

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The answer to both A and B is yes if G is abelian (and torsion-free). Question A should be true for free groups.

However, Questions A and B are incompatible for a torsion-free nonabelian group which is amenable. Which one fails? (Maybe both!)

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Example.

Let G be the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$, where $\mathbb{Z} = \langle v \rangle$ acts on $\mathbb{Z} = \langle u \rangle$ by inversion ($vuv^{-1} = u^{-1}$).

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G has an abelian subgroup of index 2 (generated by u and v^2). Thus, by the Mackey Machine, all irreducible representations of G are of dimension ≤ 2, and they can all be written down. For $0 \leq s, t \leq 2$, let

$$\pi_{s,t}(u) = \begin{bmatrix} e^{\pi is} & 0 \\ 0 & e^{-\pi is} \end{bmatrix}, \quad \pi_{s,t}(v) = \begin{bmatrix} 0 & e^{\pi it} \\ e^{\pi it} & 0 \end{bmatrix}$$

Changing basis, these become

$$\pi_{s,t}(u) = \begin{bmatrix} \cos \pi s & i \sin \pi s \\ i \sin \pi s & \cos \pi s \end{bmatrix}, \quad \pi_{s,t}(v) = \begin{bmatrix} e^{\pi it} & 0 \\ 0 & -e^{\pi it} \end{bmatrix}$$

$\pi_{1+s,t} \sim \pi(1-s,t)$ via conjugation by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so only need $0 \leq s \leq 1$.

$\pi_{s,t+1} \sim \pi(s,t)$ via conjugation by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so only need $0 \leq t \leq 1$.

If $0 < s < 1$, $\pi_{s,t}$ is irreducible; if $s = 0, 1$, it is a sum of two 1-dimensional representations.

So $C^*(G) = C_r^*(G)$ is isomorphic to the set of continuous functions f from $[0,1]^2$ to $\mathbb{M}_2$ such that

(1) $f(0,t)$ and $f(1,t)$ are diagonal for all t

(2) $f(s,1) = ad\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \cdot f(s,0)$ for all s.

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The primitive ideal space of $C^*(G)$ is thus a cylinder with points on the end circles doubled (non-Hausdorff). But the joining of top and bottom has a twist, so there is only one circle at each end, going twice around.

The C*-subalgebra B of functions which are scalars at the left and right endpoints is isomorphic to $C(\mathbb{T}) \otimes D$, where D is the 2×2 dimension drop algebra

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The unitary u is in this subalgebra B, and is $1 \otimes w$, where w is a generator for $K_1(D) \cong \mathbb{Z}_2$. Thus u^2 is in the connected component of $U(C^*(G))$, and Question A fails for G.

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The exact kernel of σ appears to be quite subtle in general.

Summarizing:

$$\begin{array}{ccccc}
G & \longrightarrow & \mathcal{U}(C^*(G)) & \longrightarrow & \mathcal{U}(C_r^*(G)) \\
\downarrow & & \downarrow & & \downarrow \\
G & \xrightarrow{\sigma} & UP(C^*(G)) & \xrightarrow{\pi_*} & UP(C_r^*(G)) \\
\downarrow & & \gamma \downarrow & & \gamma_r \downarrow \\
G/[G,G] & \longrightarrow & K_1(C^*(G)) & \xrightarrow{\pi_*} & K_1(C_r^*(G))
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The diagram commutes, and the composite map across the bottom row is injective for reasonable G by Baum-Connes.

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\end{array}$$

The diagram commutes, and the composite map across the bottom row is injective for reasonable G by Baum-Connes.

If G is K-amenable, the π_* on the bottom row is an isomorphism. There is no obvious reason why π_* in the bottom row cannot be an isomorphism even if G is not K-amenable.

The composite map $\pi_* \circ \sigma$ in the middle row is not injective in general.