

Noncommutative PL-Topology

Bruce Blackadar

Happy 65th, Eberhard!

Piecewise-Linear Topology

PL-Topology is the study of topological spaces via triangulations and piecewise-linear maps.

For us, “space” means “compact metrizable space.”

All C^* -algebras will be assumed separable.

Definition:

A *triangulation* of a space X is a homeomorphism from X to the underlying space of a (finite) simplicial complex.

Definition:

A *piecewise-linear map* between simplicial complexes is a continuous map between the underlying spaces which is linear (affine) on each subsimplex.

Triangulations are related to open covers.

If $\mathcal{U}$ is a (finite) open cover of X , the *nerve* of $\mathcal{U}$ is the simplicial complex whose vertices are the open sets in $\mathcal{U}$, with the subsimplex spanned by $U_{i_1}, \dots, U_{i_m}$ included if and only if

$$U_{i_1} \cap \dots \cap U_{i_m} \neq \emptyset .$$

Triangulations are related to open covers.

If $\mathcal{U}$ is a (finite) open cover of X , the *nerve* of $\mathcal{U}$ is the simplicial complex whose vertices are the open sets in $\mathcal{U}$, with the subsimplex spanned by $U_{i_1}, \dots, U_{i_m}$ included if and only if

$$U_{i_1} \cap \dots \cap U_{i_m} \neq \emptyset .$$

There is no natural map from X to the nerve of $\mathcal{U}$.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is no natural map from the nerve of $\mathcal{V}$ to the nerve of $\mathcal{U}$. However, there is a unique well-defined PL-homotopy class of piecewise-linear maps.

The nerves of fine open covers carry the essential homotopy information about X (Čech cohomology).

We want a finer object which gives actual maps.

Partitions of Unity

Definition:

A *partition of unity* on a space X is a set

$$\{f_1, \dots, f_n\}$$

of continuous functions from X to $[0, 1]$ such that

$$\sum_{k=1}^n f_k(x) = 1$$

for all $x \in X$.

We will assume that each f_k takes the value 1 somewhere (nondegeneracy).

A triangulation gives a partition of unity using coordinates. More generally, any continuous function from X to a simplicial complex gives a partition of unity, provided all vertices are in the range.

Conversely, a partition of unity $\mathcal{P} = \{f_1, \dots, f_n\}$ gives an open cover

$$\mathcal{U}_{\mathcal{P}} = \{U_1, \dots, U_n\}$$

where

$$U_k = \{x \in X : f_k(x) > 0\} .$$

The nondegeneracy condition says that this is a minimal open cover.

There is then a continuous function $\gamma_{\mathcal{P}}$ from X to the nerve of $\mathcal{U}_{\mathcal{P}}$ defined by sending $x \in X$ to the point with coordinates

$$(f_1(x), \dots, f_n(x)) .$$

There is thus a natural one-one correspondence between partitions of unity on X and *weak triangulations* of X : continuous functions from X to a simplicial complex for which all vertices are in the range.

Refinement of Partitions of Unity

Definition:

If $\mathcal{P} = \{f_1, \dots, f_n\}$ and $\mathcal{Q} = \{g_1, \dots, g_m\}$ are partitions of unity on X , then $\mathcal{Q}$ *refines* $\mathcal{P}$ if there are scalars α_{ij} such that

$$f_i = \sum_{j=1}^m \alpha_{ij} g_j$$

for all i .

The α_{ij} are necessarily in $[0, 1]$, and for each j we have

$$\sum_{i=1}^n \alpha_{ij} = 1 .$$

However, we rarely have

$$\sum_{j=1}^m \alpha_{ij} = 1$$

for fixed i .

However, we rarely have

$$\sum_{j=1}^m \alpha_{ij} = 1$$

for fixed i .

If $\mathcal{Q}$ refines $\mathcal{P}$, then the open cover $\mathcal{U}_{\mathcal{Q}}$ refines the open cover $\mathcal{U}_{\mathcal{P}}$, and there is a natural PL-map $\gamma_{\mathcal{Q}\mathcal{P}}$ from the nerve of $\mathcal{U}_{\mathcal{Q}}$ to the nerve of $\mathcal{U}_{\mathcal{P}}$ defined by

$$\gamma_{\mathcal{Q}\mathcal{P}}(\lambda_1, \dots, \lambda_m) = \left(\sum_{j=1}^m \alpha_{1j} \lambda_j, \dots, \sum_{j=1}^m \alpha_{nj} \lambda_j \right) .$$

These maps satisfy

$$\gamma_{\mathcal{P}} = \gamma_{\mathcal{Q}\mathcal{P}} \circ \gamma_{\mathcal{Q}}$$

as maps from X to the nerve of $\mathcal{U}_{\mathcal{P}}$.

PL-Structures on a Space

Definition

A PL-structure on a space X is a sequence $(\mathcal{P}_n)$ of partitions of unity on X , each refining the previous one, such that the sequence of corresponding open covers $\mathcal{U}_n$ eventually refines any open cover of X .

Equivalently, $\bigcup \mathcal{U}_n$ is a base for the topology of X .

It is not obvious that such a PL-structure exists on a given X . There are various ways to prove this; one of the best (in my humble opinion) is using our theorem.

If $(\mathcal{P}_n)$ is a PL-structure on X , then as above we get continuous maps γ_n from X to the nerve X_n of $\mathcal{U}_n$, and for $n < m$ a PL-map γ_{mn} from X_m to X_n satisfying

$$\gamma_n = \gamma_{mn} \circ \gamma_m .$$

Thus an inverse system of simplicial complexes (polyhedra) is obtained, along with a map from X to the inverse limit.

Theorem:

This map from X to the inverse limit is a homeomorphism. So

$$X \cong \varprojlim (X_n, \gamma_{mn}) .$$

Note that this says a little more than that X can be written as an inverse limit of polyhedra, since under the connecting maps all vertices at each stage must be in the range.

If f is a continuous function from X to $\mathbb{R}$ or $\mathbb{C}$, then f is “approximately PL”: it is a uniform limit of functions of the form $\psi \circ \gamma_n$, where $\psi : X_n \rightarrow \mathbb{C}$ is PL. [Choose n large enough that f is approximately constant on the sets of $\mathcal{U}_n$.]

The Operator Algebra Perspective

Key Observation: A partition of unity on X is just a set of positive elements of $C(X)$ of norm 1 adding to the constant function 1.

Such a set $\{f_1, \dots, f_n\}$ defines a unital (complete) order embedding $\beta_{\mathcal{P}}$ of $\mathbb{C}^n$ into $C(X)$:

$$(\lambda_1, \dots, \lambda_n) \mapsto \sum_{k=1}^n \lambda_k f_k .$$

Conversely, if β is a unital (complete) order embedding of $\mathbb{C}^n$ into $C(X)$, and $f_k = \beta(\mathbf{e}_k)$, then $\{f_1, \dots, f_n\}$ is a partition of unity on X .

If $\mathcal{P} = \{f_1, \dots, f_n\}$ is a partition of unity on X , there is also a homomorphism $\alpha_{\mathcal{P}}$ from $C(X)$ to $\mathbb{C}^n$ such that $\alpha_{\mathcal{P}} \circ \beta_{\mathcal{P}}$ is the identity on $\mathbb{C}^n$: for each k choose x_k for which $f_k(x_k) = 1$, and set

$$\alpha_{\mathcal{P}}(f) = (f(x_1), \dots, f(x_n)) .$$

This homomorphism is canonical if the partition of unity gives a true triangulation (or if $\gamma_{\mathcal{P}}$ is just injective), but requires choices in general.

If $\mathcal{Q} = \{g_1, \dots, g_m\}$ is a partition of unity refining $\mathcal{P}$, there is also a unital (complete) order embedding $\beta_{\mathcal{P}\mathcal{Q}}$ of $\mathbb{C}^n$ into $\mathbb{C}^m$ defined similarly, and

$$\beta_{\mathcal{P}} = \beta_{\mathcal{Q}} \circ \beta_{\mathcal{P}\mathcal{Q}}$$

and a homomorphism $\alpha_{\mathcal{P}\mathcal{Q}} : \mathbb{C}^m \rightarrow \mathbb{C}^n$ with $\alpha_{\mathcal{P}\mathcal{Q}} \circ \beta_{\mathcal{P}\mathcal{Q}}$ the identity on $\mathbb{C}^n$.

The $\alpha_{\mathcal{P}}$, $\alpha_{\mathcal{Q}}$, and $\alpha_{\mathcal{P}\mathcal{Q}}$ can be chosen so that

$$\alpha_{\mathcal{P}} = \alpha_{\mathcal{P}\mathcal{Q}} \circ \alpha_{\mathcal{Q}} .$$

Thus, if X is a compact metrizable space, one can generate a system

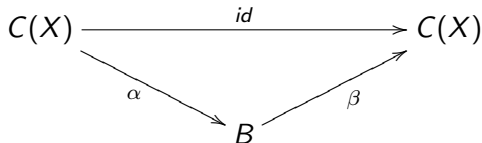
$$B_1 \rightarrow B_2 \rightarrow \dots$$

of finite-dimensional commutative C^* -algebras, where the connecting maps β_{nm} are not homomorphisms but are complete order embeddings, and compatible complete order embeddings

$$\beta_n : B_n \rightarrow C(X)$$

such that the union $\cup_n \beta_n(B_n)$ is dense in $C(X)$, and (unique!) homomorphisms $\alpha_n : C(X) \rightarrow B_n$ which are coherent and left inverses for the β_n .

It is notationally convenient to write things “locally” by saying there are diagrams

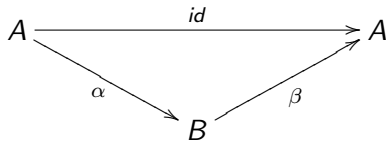


which approximately commute in the point-norm topology, where the B is finite-dimensional and α and β are completely positive contractions, with the additional properties that

- (1) α is a homomorphism.
- (2) $\alpha \circ \beta$ is the identity on B . (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)
- (3) β is a complete order embedding.

The Noncommutative Case

If A is a (separable) C^* -algebra, it is natural to regard an inductive system of finite-dimensional C^* -algebras and complete order embeddings into A with analogous properties to be a “PL structure” on A . Phrasing things locally, we want a set of diagrams



which approximately commute in the point-norm topology, where the B is finite-dimensional and α and β are completely positive contractions, satisfying as many of (1)–(3) as possible.

One can pass to the inductive system picture by fairly routine perturbation arguments (if A is separable).

To get started, we need A to be nuclear.

In addition, we want at least some of the following:

- (1) α is a homomorphism.
- (2) $\alpha \circ \beta$ is the identity on B . (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)
- (3) β is a complete order embedding.

To get started, we need A to be nuclear.

In addition, we want at least some of the following:

- (1) α is a homomorphism.
- (2) $\alpha \circ \beta$ is the identity on B . (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)
- (3) β is a complete order embedding.

We can only hope for (1) if A is residually finite-dimensional.

Theorem:

If A is nuclear and residually finite-dimensional, we can get (1)–(3). (Such a C^* -algebra is called an *RF algebra*.)

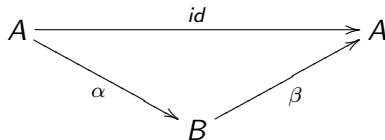
We can, however, weaken (1) to
(4) α is approximately multiplicative.

Condition (4) is extremely natural: it means that not only the complete order structure but also the algebraic structure (multiplication) of A can be approximated in finite-dimensional C^* -algebras.

We can only hope to get (4) if A is stably finite.

Definition:

A separable C^* -algebra A is an *NF algebra* if, for any $x_1, \dots, x_n \in A$ and $\epsilon > 0$ there is a finite-dimensional C^* -algebra B and completely positive contractions $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$ such that $\|\beta \circ \alpha(x_i) - x_i\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all i, j .



Here are a few of the many characterizations of NF algebras:

Theorem:

Let A be a separable C^* -algebra. The following are equivalent:

- (i) A is an NF algebra.
- (ii) A is nuclear and quasidiagonal.
- (iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C^* -algebras in which the connecting maps are completely positive contractions (and asymptotically multiplicative).

Here are a few of the many characterizations of NF algebras:

Theorem:

Let A be a separable C^* -algebra. The following are equivalent:

- (i) A is an NF algebra.
- (ii) A is nuclear and quasidiagonal.
- (iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C^* -algebras in which the connecting maps are completely positive contractions (and asymptotically multiplicative).

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \dots \longrightarrow A$$

Such a system is called an *NF system* for A .

If $(A_n, \phi_{n,n+1})$ is an NF system for A , there is a completely positive contraction $\phi_n : A_n \rightarrow A$, and $\cup_n \phi_n(A_n)$ is dense in A . But $\phi_n(A_n)$ is not a subalgebra of A in general.

An NF system gives a “combinatorial” description of A . The study of NF algebras via NF systems can be called “noncommutative PL topology.”

If $(A_n, \phi_{n,n+1})$ is an NF system for A , there is a completely positive contraction $\phi_n : A_n \rightarrow A$, and $\cup_n \phi_n(A_n)$ is dense in A . But $\phi_n(A_n)$ is not a subalgebra of A in general.

An NF system gives a “combinatorial” description of A . The study of NF algebras via NF systems can be called “noncommutative PL topology.”

From the quasidiagonality characterization, we obtain the fact, not obvious from the definition, that a nuclear C^* -subalgebra of an NF algebra is NF.

A quotient of an NF algebra is not necessarily NF. In fact, any separable nuclear C^* -algebra is a quotient of an NF algebra [if A is separable and nuclear, then the cone over A is an NF algebra by Voiculescu.]

Conditions (2) and (3) are closely related: diagrams satisfying (2) automatically satisfy (3), and diagrams satisfying (3) can be modified to diagrams satisfying (2). Diagrams satisfying (2) automatically satisfy (4).

Conditions (2) and (3) are closely related: diagrams satisfying (2) automatically satisfy (3), and diagrams satisfying (3) can be modified to diagrams satisfying (2). Diagrams satisfying (2) automatically satisfy (4).

It turns out that we cannot always get (2) or (3) for NF algebras.

Definition:

A (separable) C^* -algebra A with diagrams satisfying (2) (hence also (3) and (4)) is a *strong NF algebra*.

Here are a few of the many characterizations of strong NF algebras:

Theorem:

Let A be a separable C^* -algebra. The following are equivalent:

- (i) A is a strong NF algebra.
- (ii) A is nuclear and has a separating family of quasidiagonal irreducible representations.
- (iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C^* -algebras in which the connecting maps are complete order embeddings (and asymptotically multiplicative).

Here are a few of the many characterizations of strong NF algebras:

Theorem:

Let A be a separable C^* -algebra. The following are equivalent:

- (i) A is a strong NF algebra.
- (ii) A is nuclear and has a separating family of quasidiagonal irreducible representations.
- (iii) A can be written as a generalized inductive limit of a sequence of finite-dimensional C^* -algebras in which the connecting maps are complete order embeddings (and asymptotically multiplicative).

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \dots \longrightarrow A$$

Such a system is called a *strong NF system* for A .

If A is a C^* -algebra and every quotient of A is an NF algebra (i.e. A is strongly quasidiagonal), then A is a strong NF algebra.

In particular, every simple NF algebra is a strong NF algebra.

Theorem:

Every strong NF algebra is an *ordinary* inductive limit of RF algebras (with injective connecting maps).

Ideal Structure of NF Algebras

Ideals (closed, two-sided) in the inductive limit of an ordinary inductive system can be read off from the system, at least in principle. But ideals in the inductive limit of a generalized inductive system are much harder to describe.

Ideal Structure of NF Algebras

Ideals (closed, two-sided) in the inductive limit of an ordinary inductive system can be read off from the system, at least in principle. But ideals in the inductive limit of a generalized inductive system are much harder to describe.

Example. Let $A_n = \mathbb{M}_n$,

$$\phi_{n,n+1} \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

The inductive limit A is isomorphic to $\mathbb{K} + \mathbb{C}1$. Each A_n is simple, but A is not simple.

But there is something that can be said about ideals in a generalized inductive limit. Let $A = \lim_{\rightarrow} (A_n, \phi_{n,n+1})$, and let J be an ideal in A . Since $\phi_n(A_n)$ is not a subalgebra of A , $\phi_n^{-1}(J)$ is not a subalgebra of A_n in general.

However, since ϕ_n is positive, $\phi_n^{-1}(J) \cap A_{n+}$ is a (closed) hereditary cone in A_{n+} , so its span is a hereditary C^* -subalgebra J_n of A_n (not an ideal in general.) Since A_n is finite-dimensional, J_n is a corner, i.e. $J_n = p_n A_n p_n$ for a projection $p_n \in A_n$.

$\phi_{n,n+1}(p_n)$ is not a projection in A_{n+1} in general. However, p_{n+1} is a unit for $\phi_{n,n+1}(p_n)$.

The closure of $\cup \phi_n(J_n)$ is an ideal of A contained in J .

The closure of $\cup \phi_n(J_n)$ is an ideal of A contained in J .

Definition:

If $\cup \phi_n(J_n)$ is dense in J , then the ideal J is *induced* from the system $(A_n, \phi_{n,n+1})$. J is an *induced ideal* of A if it is induced from some NF system for A .

It is unclear whether an ideal can be induced from one NF system but not from another.

Proposition:

In $C(X)$, every ideal is induced from any NF system.

To see why, suppose J is an ideal of $C(X)$ consisting of all functions vanishing on a closed set Y in X . A_n consists of all piecewise-linear functions on a simplicial complex X_n , with $\gamma_n : X \rightarrow X_n$ a continuous map whose range contains the vertices of X_n .

Then J_n consists of the span of the set of nonnegative piecewise-linear functions vanishing on $\gamma_n(Y)$ (an ideal in A_n in this case). Such a function also vanishes on any entire subsimplex of X_n containing a point of $\gamma_n(Y)$ in its interior. $\phi_n(J_n)$ consists of all functions vanishing on the inverse image Y_n under γ_n of these simplexes.

We have $Y \subseteq Y_n$ for each n . If ρ is a metric on X and $\epsilon > 0$, then for sufficiently large n we have Y_n contained in an ϵ -neighborhood of Y . Thus $\cap Y_n = Y$ and $\cup_n \phi_n(J_n)$ is dense in J .

Not every ideal in an NF algebra is induced:

Not every ideal in an NF algebra is induced:

Definition:

An ideal J in an NF algebra A is an *NF ideal* if A/J is an NF algebra.

Not every ideal in an NF algebra is an NF ideal.

Proposition:

An induced ideal is an NF ideal.

Proposition:

An induced ideal is an NF ideal.

Suppose J is induced from the system $(A_n, \phi_{n,n+1})$, and let p_n be the projection in A_n corresponding to J_n . Define a generalized inductive system as follows: let $B_n = (1 - p_n)A_n(1 - p_n)$, and define $\psi_{n,n+1} : B_n \rightarrow B_{n+1}$ by

$$\psi_{n,n+1}(x) = (1 - p_{n+1})\phi_{n,n+1}(x)(1 - p_{n+1}).$$

It is routine to check that this is indeed a generalized inductive system, hence an NF system, and that the generalized inductive limit is naturally isomorphic to A/J .

Using a similar (but simplified) argument, one can show:

Proposition:

Let J be an ideal in an NF algebra A . If J has a quasicentral approximate unit of projections, then J is an NF ideal.

This can also be proved using the known result that the quotient of a quasidiagonal C^* -algebra by an ideal with a quasicentral approximate unit of projections is quasidiagonal.

Using a similar (but simplified) argument, one can show:

Proposition:

Let J be an ideal in an NF algebra A . If J has a quasicentral approximate unit of projections, then J is an NF ideal.

This can also be proved using the known result that the quotient of a quasidiagonal C^* -algebra by an ideal with a quasicentral approximate unit of projections is quasidiagonal.

We say an ideal J in a C^* -algebra A is *locally approximately split* if, for every $x_1, \dots, x_n \in A/J$ and $\epsilon > 0$, there is a completely positive contraction $\sigma : A/J \rightarrow A$ such that $\|\pi \circ \sigma(x_i) - x_i\| < \epsilon$ and $\|\sigma(x_i x_j) - \sigma(x_i)\sigma(x_j)\| < \epsilon$ for all i, j , where $\pi : A \rightarrow A/J$ is the quotient map.

Using a similar (but simplified) argument, one can show:

Proposition:

Let J be an ideal in an NF algebra A . If J has a quasicentral approximate unit of projections, then J is an NF ideal.

This can also be proved using the known result that the quotient of a quasidiagonal C^* -algebra by an ideal with a quasicentral approximate unit of projections is quasidiagonal.

We say an ideal J in a C^* -algebra A is *locally approximately split* if, for every $x_1, \dots, x_n \in A/J$ and $\epsilon > 0$, there is a completely positive contraction $\sigma : A/J \rightarrow A$ such that $\|\pi \circ \sigma(x_i) - x_i\| < \epsilon$ and $\|\sigma(x_i x_j) - \sigma(x_i)\sigma(x_j)\| < \epsilon$ for all i, j , where $\pi : A \rightarrow A/J$ is the quotient map.

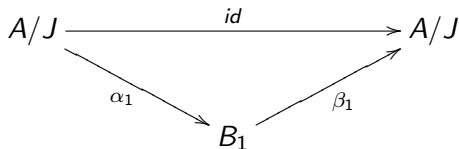
Proposition:

A locally approximately split ideal in an NF algebra is an NF ideal.

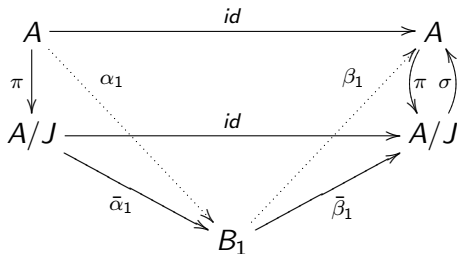
There is a “converse” to the proposition. Suppose J is an NF ideal in an NF algebra A . We want to show that J is an induced ideal from some NF system for A . We outline the argument.

There is a “converse” to the proposition. Suppose J is an NF ideal in an NF algebra A . We want to show that J is an induced ideal from some NF system for A . We outline the argument.

Begin with finite subsets $\{x_i\}$ of A and $\{y_j\}$ of J and an approximately commutative diagram



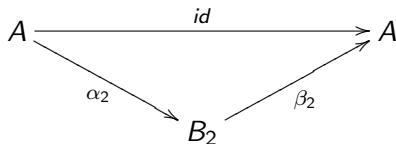
Lift the diagram to



This diagram only approximately commutes mod J .

Using a quascentral approximate unit for J , choose $h \in J_+$, $\|h\| \leq 1$, such that h almost commutes with the elements of A and is almost a unit for the elements of J .

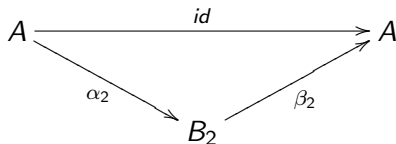
Choose a diagram



which is approximately commutative and approximately multiplicative on all elements defined so far.

Using a quasicentral approximate unit for J , choose $h \in J_+$, $\|h\| \leq 1$, such that h almost commutes with the elements of A and is almost a unit for the elements of J .

Choose a diagram



which is approximately commutative and approximately multiplicative on all elements defined so far.

Let $\alpha = \alpha_1 \oplus \alpha_2 : A \rightarrow B_1 \oplus B_2$ and $\beta : B_1 \oplus B_2 \rightarrow A$, where

$$\beta(x, y) = (1 - h)^{1/2} \beta_1(x) (1 - h)^{1/2} + h^{1/2} \beta_2(y) h^{1/2}$$

To get an NF system, set $A_1 = B_1 \oplus B_2$. At the next stage, reduce ϵ and expand $\{x_i\}$ by throwing in all the images of matrix units of $B_1 \oplus B_2$ and more elements of a dense subset of A , and expand $\{y_i\}$ by throwing in the images of the matrix units of B_2 (which lie in J) as well as more elements of a dense subset of J .

The ideal J is induced by this NF system.

So the conclusion is:

Theorem:

Let A be an NF algebra, J an ideal of A . Then J is an NF ideal if and only if J is induced by some NF system for A .

It would be nice to get a single NF system for an NF algebra A such that all NF ideals can be induced from this system. It is really only necessary to be able to do the previous construction with two NF ideals J and K (or finitely many) simultaneously.

If $J \cap K$ and $J + K$ are also NF ideals, it appears the construction can be made to work with some technical complications. It is true that $J \cap K$ is always an NF ideal, since $A/(J \cap K)$ can be embedded in $A/J \oplus A/K$ and hence is an NF algebra. But it is not obvious that $J + K$ is always NF.

The construction at least appears to work for residually NF algebras (strongly quasidiagonal nuclear C^* -algebras).

A potential application is to the following fundamental question:

Question:

Is every stably finite separable nuclear C^* -algebra an NF algebra?

A potential application is to the following fundamental question:

Question:

Is every stably finite separable nuclear C^* -algebra an NF algebra?

The theorem gives a potential approach to showing that a separable nuclear C^* -algebra B with a faithful tracial state τ must be an NF algebra. Any separable nuclear C^* -algebra is a quotient of an NF algebra, so B is a quotient of an NF algebra A , and τ may be regarded as a tracial state on A . If J is the kernel of τ , i.e.

$$J = \{x \in A : \tau(x^*x) = 0\}$$

then J is the kernel of the quotient map from A to B . So it suffices to show that the kernel of a tracial state on an NF algebra is an induced ideal.