# Noncommutative PL-Topology

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Happy 65th, Eberhard!

## Piecewise-Linear Topology

PL-Topology is the study of topological spaces via triangulations and piecewise-linear maps.

For us, "space" means "compact metrizable space."

All C*-algebras will be assumed separable.

**Definition:**

A *triangulation* of a space $X$ is a homeomorphism from $X$ to the underlying space of a (finite) simplicial complex.

**Definition:**

A *piecewise-linear map* between simplicial complexes is a continuous map between the underlying spaces which is linear (affine) on each subsimplex.

Triangulations are related to open covers.

If $\mathcal{U}$ is a (finite) open cover of $X$, the *nerve* of $\mathcal{U}$ is the simplicial complex whose vertices are the open sets in $\mathcal{U}$, with the subsimplex spanned by $U_{i_1}, \ldots, U_{i_m}$ included if and only if

$$U_{i_1} \cap \cdots \cap U_{i_m} \neq \emptyset \ .$$

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$$U_{i_1} \cap \cdots \cap U_{i_m} \neq \emptyset \ .$$

There is no natural map from $X$ to the nerve of $\mathcal{U}$.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is no natural map from the nerve of $\mathcal{V}$ to the nerve of $\mathcal{U}$. However, there is a unique well-defined PL-homotopy class of piecewise-linear maps.

The nerves of fine open covers carry the essential homotopy information about $X$ (Čech cohomology).

We want a finer object which gives actual maps.

## Partitions of Unity

**Definition:**

A *partition of unity* on a space $X$ is a set

$$\{f_1, \dots, f_n\}$$

of continuous functions from $X$ to $[0, 1]$ such that

$$\sum_{k=1}^{n} f_k(x) = 1$$

for all $x \in X$.

We will assume that each $f_k$ takes the value 1 somewhere (nondegeneracy).

A triangulation gives a partition of unity using coordinates. More generally, any continuous function from $X$ to a simplicial complex gives a partition of unity, provided all vertices are in the range.

Conversely, a partition of unity $\mathcal{P} = \{f_1, \ldots, f_n\}$ gives an open cover

$$\mathcal{U}_{\mathcal{P}} = \{U_1, \ldots, U_n\}$$

where

$$U_k = \{x \in X : f_k(x) > 0\} \ .$$

The nondegeneracy condition says that this is a minimal open cover.

There is then a continuous function $\gamma_{\mathcal{P}}$ from $X$ to the nerve of $\mathcal{U}_{\mathcal{P}}$ defined by sending $x \in X$ to the point with coordinates

$$(f_1(x), \ldots, f_n(x)) .$$

There is thus a natural one-one correspondence between partitions of unity on $X$ and *weak triangulations* of $X$: continuous functions from $X$ to a simplicial complex for which all vertices are in the range.

## Refinement of Partitions of Unity

**Definition:**

If $\mathcal{P} = \{f_1, \ldots, f_n\}$ and $\mathcal{Q} = \{g_1, \ldots, g_m\}$ are partitions of unity on $X$, then $\mathcal{Q}$ *refines* $\mathcal{P}$ if there are scalars $\alpha_{ij}$ such that

$$f_i = \sum_{j=1}^{m} \alpha_{ij} g_j$$

for all $i$.

The $\alpha_{ij}$ are necessarily in $[0, 1]$, and for each $j$ we have

$$\sum_{i=1}^{n} \alpha_{ij} = 1 \ .$$

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If $\mathcal{Q}$ refines $\mathcal{P}$, then the open cover $\mathcal{U}_\mathcal{Q}$ refines the open cover $\mathcal{U}_\mathcal{P}$, and there is a natural PL-map $\gamma_{\mathcal{Q}\mathcal{P}}$ from the nerve of $\mathcal{U}_\mathcal{Q}$ to the nerve of $\mathcal{U}_\mathcal{P}$ defined by

$$\gamma_{\mathcal{Q}\mathcal{P}}(\lambda_1, \ldots, \lambda_m) = \left( \sum_{j=1}^{m} \alpha_{1j}\lambda_j, \ldots, \sum_{j=1}^{m} \alpha_{nj}\lambda_j \right) .$$

These maps satisfy

$$\gamma_\mathcal{P} = \gamma_{\mathcal{Q}\mathcal{P}} \circ \gamma_\mathcal{Q}$$

as maps from $X$ to the nerve of $\mathcal{U}_\mathcal{P}$.

## PL-Structures on a Space

**Definition**

A PL-structure on a space $X$ is a sequence $(\mathcal{P}_n)$ of partitions of unity on $X$, each refining the previous one, such that the sequence of corresponding open covers $\mathcal{U}_n$ eventually refines any open cover of $X$.

Equivalently, $\cup\mathcal{U}_n$ is a base for the topology of $X$.

It is not obvious that such a PL-structure exists on a given $X$. There are various ways to prove this; one of the best (in my humble opinion) is using our theorem.

If $(\mathcal{P}_n)$ is a PL-structure on $X$, then as above we get continuous maps $\gamma_n$ from $X$ to the nerve $X_n$ of $\mathcal{U}_n$, and for $n < m$ a PL-map $\gamma_{mn}$ from $X_m$ to $X_n$ satisfying

$$\gamma_n = \gamma_{mn} \circ \gamma_m \ .$$

Thus an inverse system of simplicial complexes (polyhedra) is obtained, along with a map from $X$ to the inverse limit.

**Theorem:**

This map from $X$ to the inverse limit is a homeomorphism. So

$$X \cong \varprojlim (X_n, \gamma_{mn}) \ .$$

Note that this says a little more than that $X$ can be written as an inverse limit of polyhedra, since under the connecting maps all vertices at each stage must be in the range.

If $f$ is a continuous function from $X$ to $\mathbb{R}$ or $\mathbb{C}$, then $f$ is "approximately PL": it is a uniform limit of functions of the form $\psi \circ \gamma_n$, where $\psi : X_n \to \mathbb{C}$ is PL. [Choose $n$ large enough that $f$ is approximately constant on the sets of $\mathcal{U}_n$.]

## The Operator Algebra Perspective

**Key Observation:** A partition of unity on $X$ is just a set of positive elements of $C(X)$ of norm 1 adding to the constant function 1.

Such a set $\{f_1, \ldots, f_n\}$ defines a unital (complete) order embedding $\beta_{\mathcal{P}}$ of $\mathbb{C}^n$ into $C(X)$:

$$(\lambda_1, \ldots, \lambda_n) \mapsto \sum_{k=1}^{n} \lambda_k f_k \ .$$

Conversely, if $\beta$ is a unital (complete) order embedding of $\mathbb{C}^n$ into $C(X)$, and $f_k = \beta(\mathbf{e}_k)$, then $\{f_1, \ldots, f_n\}$ is a partition of unity on $X$.

If $\mathcal{P} = \{f_1, \ldots, f_n\}$ is a partition of unity on $X$, there is also a homomorphism $\alpha_{\mathcal{P}}$ from $C(X)$ to $\mathbb{C}^n$ such that $\alpha_{\mathcal{P}} \circ \beta_{\mathcal{P}}$ is the identity on $\mathbb{C}^n$: for each $k$ choose $x_k$ for which $f_k(x_k) = 1$, and set

$$\alpha_{\mathcal{P}}(f) = (f(x_1), \ldots, f(x_n)) \ .$$

This homomorphism is canonical if the partition of unity gives a true triangulation (or if $\gamma_{\mathcal{P}}$ is just injective), but requires choices in general.

If $\mathcal{Q} = \{g_1, \ldots, g_m\}$ is a partition of unity refining $\mathcal{P}$, there is also a unital (complete) order embedding $\beta_{\mathcal{P}\mathcal{Q}}$ of $\mathbb{C}^n$ into $\mathbb{C}^m$ defined similarly, and

$$\beta_{\mathcal{P}} = \beta_{\mathcal{Q}} \circ \beta_{\mathcal{P}\mathcal{Q}}$$

and a homomorphism $\alpha_{\mathcal{P}\mathcal{Q}} : \mathbb{C}^m \to \mathbb{C}^n$ with $\alpha_{\mathcal{P}\mathcal{Q}} \circ \beta_{\mathcal{P}\mathcal{Q}}$ the identity on $\mathbb{C}^n$.

The $\alpha_{\mathcal{P}}$, $\alpha_{\mathcal{Q}}$, and $\alpha_{\mathcal{P}\mathcal{Q}}$ can be chosen so that

$$\alpha_{\mathcal{P}} = \alpha_{\mathcal{P}\mathcal{Q}} \circ \alpha_{\mathcal{Q}} \ .$$

Thus, if $X$ is a compact metrizable space, one can generate a system

$$B_1 \to B_2 \to \cdots$$

of finite-dimensional commutative C*-algebras, where the connecting maps $\beta_{nm}$ are not homomorphisms but are complete order embeddings, and compatible complete order embeddings

$$\beta_n : B_n \to C(X)$$

such that the union $\cup_n \beta_n(B_n)$ is dense in $C(X)$, and (unique!) homomorphisms $\alpha_n : C(X) \to B_n$ which are coherent and left inverses for the $\beta_n$.

It is notationally convenient to write things "locally" by saying there are diagrams

$$\begin{tikzcd} C(X) \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & C(X) \\ & B \arrow[ur, "\beta"'] & \end{tikzcd}$$

which approximately commute in the point-norm topology, where the $B$ is finite-dimensional and $\alpha$ and $\beta$ are completely positive contractions, with the additional properties that
(1) $\alpha$ is a homomorphism.
(2) $\alpha \circ \beta$ is the identity on $B$. (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)
(3) $\beta$ is a complete order embedding.

## The Noncommutative Case

If $A$ is a (separable) C*-algebra, it is natural to regard an inductive system of finite-dimensional C*-algebras and complete order embeddings into $A$ with analogous properties to be a “PL structure” on $A$. Phrasing things locally, we want a set of diagrams

$$\begin{tikzcd}
A \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & A \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

which approximately commute in the point-norm topology, where the $B$ is finite-dimensional and $\alpha$ and $\beta$ are completely positive contractions, satisfying as many of (1)–(3) as possible.

One can pass to the inductive system picture by fairly routine perturbation arguments (if $A$ is separable).

To get started, we need $A$ to be nuclear.

In addition, we want at least some of the following:

(1) $\alpha$ is a homomorphism.

(2) $\alpha \circ \beta$ is the identity on $B$. (Hence $\beta \circ \alpha$ is an idempotent map from $C(X)$ to $C(X)$.)

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(3) $\beta$ is a complete order embedding.

We can only hope for (1) if $A$ is residually finite-dimensional.

**Theorem:**

If $A$ is nuclear and residually finite-dimensional, we can get (1)–(3). (Such a C*-algebra is called an *RF algebra*.)

We can, however, weaken (1) to

(4) $\alpha$ is approximately multiplicative.

Condition (4) is extremely natural: it means that not only the complete order structure but also the algebraic structure (multiplication) of $A$ can be approximated in finite-dimensional C*-algebras.

We can only hope to get (4) if $A$ is stably finite.

## Definition:

A separable C*-algebra $A$ is an *NF algebra* if, for any $x_1, \ldots, x_n \in A$ and $\epsilon > 0$ there is a finite-dimensional C*-algebra $B$ and completely positive contractions $\alpha : A \to B$ and $\beta : B \to A$ such that $\|\beta \circ \alpha(x_i) - x_i\| < \epsilon$ and $\|\alpha(x_i x_j) - \alpha(x_i)\alpha(x_j)\| < \epsilon$ for all $i, j$.

$$\begin{tikzcd}
A \arrow[rr, "id"] \arrow[dr, "\alpha"'] & & A \\
& B \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Here are a few of the many characterizations of NF algebras:

**Theorem:**

Let $A$ be a separable C*-algebra. The following are equivalent:

(i) $A$ is an NF algebra.

(ii) $A$ is nuclear and quasidiagonal.

(iii) $A$ can be written as a generalized inductive limit of a sequence of finite-dimensional C*-algebras in which the connecting maps are completely positive contractions (and asymptotically multiplicative).

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$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \longrightarrow A$$

Such a system is called an *NF system* for $A$.

If $(A_n, \phi_{n,n+1})$ is an NF system for $A$, there is a completely positive contraction $\phi_n : A_n \to A$, and $\cup_n \phi_n(A_n)$ is dense in $A$. But $\phi_n(A_n)$ is not a subalgebra of $A$ in general.

An NF system gives a "combinatorial" description of $A$. The study of NF algebras via NF systems can be called "noncommutative PL topology."

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From the quasidiagonality characterization, we obtain the fact, not obvious from the definition, that a nuclear C*-subalgebra of an NF algebra is NF.

A quotient of an NF algebra is not necessarily NF. In fact, any separable nuclear C*-algebra is a quotient of an NF algebra [if $A$ is separable and nuclear, then the cone over $A$ is an NF algebra by Voiculescu.]

Conditions (2) and (3) are closely related: diagrams satisfying (2) automatically satisfy (3), and diagrams satisfying (3) can be modified to diagrams satisfying (2). Diagrams satisfying (2) automatically satisfy (4).

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It turns out that we cannot always get (2) or (3) for NF algebras.

**Definition:**

A (separable) C*-algebra $A$ with diagrams satisfying (2) (hence also (3) and (4)) is a *strong NF algebra*.

Here are a few of the many characterizations of strong NF algebras:

**Theorem:**

Let $A$ be a separable C*-algebra. The following are equivalent:

(i) $A$ is a strong NF algebra.

(ii) $A$ is nuclear and has a separating family of quasidiagonal irreducible representations.

(iii) $A$ can be written as a generalized inductive limit of a sequence of finite-dimensional C*-algebras in which the connecting maps are complete order embeddings (and asymptotically multiplicative).

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$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \xrightarrow{\phi_{3,4}} \cdots \longrightarrow A$$

Such a system is called a *strong NF system* for $A$.

If $A$ is a C*-algebra and every quotient of $A$ is an NF algebra (i.e. $A$ is strongly quasidiagonal), then $A$ is a strong NF algebra.

In particular, every simple NF algebra is a strong NF algebra.

**Theorem:**

Every strong NF algebra is an *ordinary* inductive limit of RF algebras (with injective connecting maps).

## Ideal Structure of NF Algebras

Ideals (closed, two-sided) in the inductive limit of an ordinary inductive system can be read off from the system, at least in principle. But ideals in the inductive limit of a generalized inductive system are much harder to describe.

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**Example.** Let $A_n = \mathbb{M}_n$,

$$\phi_{n,n+1}\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

The inductive limit $A$ is isomorphic to $\mathbb{K} + \mathbb{C}1$. Each $A_n$ is simple, but $A$ is not simple.

But there is something that can be said about ideals in a generalized inductive limit. Let $A = \lim_{\rightarrow}(A_n, \phi_{n,n+1})$, and let $J$ be an ideal in $A$. Since $\phi_n(A_n)$ is not a subalgebra of $A$, $\phi_n^{-1}(J)$ is not a subalgebra of $A_n$ in general.

However, since $\phi_n$ is positive, $\phi_n^{-1}(J) \cap A_{n+}$ is a (closed) hereditary cone in $A_{n+}$, so its span is a hereditary C*-subalgebra $J_n$ of $A_n$ (not an ideal in general.) Since $A_n$ is finite-dimensional, $J_n$ is a corner, i.e. $J_n = p_n A_n p_n$ for a projection $p_n \in A_n$.

$\phi_{n,n+1}(p_n)$ is not a projection in $A_{n+1}$ in general. However, $p_{n+1}$ is a unit for $\phi_{n,n+1}(p_n)$.

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**Definition:**

If $\cup\phi_n(J_n)$ is dense in $J$, then the ideal $J$ is *induced* from the system $(A_n, \phi_{n,n+1})$. $J$ is an *induced ideal* of $A$ if it is induced from some NF system for $A$.

It is unclear whether an ideal can be induced from one NF system but not from another.

**Proposition:**

In $C(X)$, every ideal is induced from any NF system.

To see why, suppose $J$ is an ideal of $C(X)$ consisting of all functions vanishing on a closed set $Y$ in $X$. $A_n$ consists of all piecewise-linear functions on a simplicial complex $X_n$, with $\gamma_n : X \to X_n$ a continuous map whose range contains the vertices of $X_n$.

Then $J_n$ consists of the span of the set of nonnegative piecewise-linear functions vanishing on $\gamma_n(Y)$ (an ideal in $A_n$ in this case). Such a function also vanishes on any entire subsimplex of $X_n$ containing a point of $\gamma_n(Y)$ in its interior. $\phi_n(J_n)$ consists of all functions vanishing on the inverse image $Y_n$ under $\gamma_n$ of these simplexes.

We have $Y \subseteq Y_n$ for each $n$. If $\rho$ is a metric on $X$ and $\epsilon > 0$, then for sufficiently large $n$ we have $Y_n$ contained in an $\epsilon$-neighborhood of $Y$. Thus $\cap Y_n = Y$ and $\cup_n \phi_n(J_n)$ is dense in $J$.

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**Definition:**

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Suppose $J$ is induced from the system $(A_n, \phi_{n,n+1})$, and let $p_n$ be the projection in $A_n$ corresponding to $J_n$. Define a generalized inductive system as follows: let $B_n = (1 - p_n)A_n(1 - p_n)$, and define $\psi_{n,n+1} : B_n \to B_{n+1}$ by

$$\psi_{n,n+1}(x) = (1 - p_{n+1})\phi_{n,n+1}(x)(1 - p_{n+1})\,.$$

It is routine to check that this is indeed a generalized inductive system, hence an NF system, and that the generalized inductive limit is naturally isomorphic to $A/J$.

Using a similar (but simplified) argument, one can show:

**Proposition:**

Let $J$ be an ideal in an NF algebra $A$. If $J$ has a quasicentral approximate unit of projections, then $J$ is an NF ideal.

This can also be proved using the known result that the quotient of a quasidiagonal C*-algebra by an ideal with a quasicentral approximate unit of projections is quasidiagonal.

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We say an ideal $J$ in a C*-algebra $A$ is *locally approximately split* if, for every $x_1, \ldots, x_n \in A/J$ and $\epsilon > 0$, there is a completely positive contraction $\sigma : A/J \to A$ such that $\|\pi \circ \sigma(x_i) - x_i\| < \epsilon$ and $\|\sigma(x_i x_j) - \sigma(x_i)\sigma(x_j)\| < \epsilon$ for all $i, j$, where $\pi : A \to A/J$ is the quotient map.

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**Proposition:**

A locally approximately split ideal in an NF algebra is an NF ideal.

There is a "converse" to the proposition. Suppose $J$ is an NF ideal in an NF algebra $A$. We want to show that $J$ is an induced ideal from some NF system for $A$. We outline the argument.

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Begin with finite subsets $\{x_i\}$ of $A$ and $\{y_j\}$ of $J$ and an approximately commutative diagram

$$\begin{tikzcd} A/J \arrow[rr, "id"] \arrow[dr, "\alpha_1"'] & & A/J \\ & B_1 \arrow[ur, "\beta_1"'] & \end{tikzcd}$$

Lift the diagram to

$$\begin{tikzcd}
A \arrow[rr, "id"] \arrow[d, "\pi"'] \arrow[ddr, dotted, "\alpha_1"] & & A \arrow[d, bend right, "\pi"'] \\
A/J \arrow[rr, "id"] \arrow[dr, "\bar{\alpha}_1"'] & & A/J \arrow[u, bend right, "\sigma"'] \\
& B_1 \arrow[ur, "\bar{\beta}_1"'] \arrow[uur, dotted, "\beta_1"] &
\end{tikzcd}$$

This diagram only approximately commutes mod $J$.

Using a quasicentral approximate unit for $J$, choose $h \in J_+$, $\|h\| \leq 1$, such that $h$ almost commutes with the elements of $A$ and is almost a unit for the elements of $J$.

Choose a diagram

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "\alpha_2"'] & & A \\ & B_2 \arrow[ur, "\beta_2"'] & \end{tikzcd}$$

which is approximately commutative and approximately multiplicative on all elements defined so far.

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which is approximately commutative and approximately multiplicative on all elements defined so far.

Let $\alpha = \alpha_1 \oplus \alpha_2 : A \to B_1 \oplus B_2$ and $\beta : B_1 \oplus B_2 \to A$, where

$$\beta(x,y) = (1-h)^{1/2}\beta_1(x)(1-h)^{1/2} + h^{1/2}\beta_2(y)h^{1/2}$$

To get an NF system, set $A_1 = B_1 \oplus B_2$. At the next stage, reduce $\epsilon$ and expand $\{x_i\}$ by throwing in all the images of matrix units of $B_1 \oplus B_2$ and more elements of a dense subset of $A$, and expand $\{y_i\}$ by throwing in the images of the matrix units of $B_2$ (which lie in $J$) as well as more elements of a dense subset of $J$.

The ideal $J$ is induced by this NF system.

So the conclusion is:

**Theorem:**

Let $A$ be an NF algebra, $J$ an ideal of $A$. Then $J$ is an NF ideal if and only if $J$ is induced by some NF system for $A$.

It would be nice to get a single NF system for an NF algebra $A$ such that all NF ideals can be induced from this system. It is really only necessary to be able to do the previous construction with two NF ideals $J$ and $K$ (or finitely many) simultaneously.

If $J \cap K$ and $J + K$ are also NF ideals, it appears the construction can be made to work with some technical complications. It is true that $J \cap K$ is always an NF ideal, since $A/(J \cap K)$ can be embedded in $A/J \oplus A/K$ and hence is an NF algebra. But it is not obvious that $J + K$ is always NF.

The construction at least appears to work for residually NF algebras (strongly quasidiagonal nuclear C*-algebras).

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The theorem gives a potential approach to showing that a separable nuclear C*-algebra $B$ with a faithful tracial state $\tau$ must be an NF algebra. Any separable nuclear C*-algebra is a quotient of an NF algebra, so $B$ is a quotient of an NF algebra $A$, and $\tau$ may be regarded as a tracial state on $A$. If $J$ is the kernel of $\tau$, i.e.

$$J = \{x \in A : \tau(x^*x) = 0\}$$

then $J$ is the kernel of the quotient map from $A$ to $B$. So it suffices to show that the kernel of a tracial state on an NF algebra is an induced ideal.