Fun With Continued Fractions

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Representing Real Numbers

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Geometric: Points on number line

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Notational: Infinite decimals

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Can use base n “decimals” for any n.

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Continued fractions are considered part of number theory, but have applications in analysis, probability, dynamical systems, topology, . . .

Definition.

A *(simple) continued fraction* is an expression of the form

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\cdots + \cfrac{1}{x_n + \cdots}}}}} .$$

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A *finite continued fraction* of depth $n \in \mathbb{N}$ is an expression of the form

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It is notationally clumsy to write these complex fractions; we will use the common notation

$$[x_0; x_1, \dots, x_n]$$

for the continued fraction

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\cdots + \cfrac{1}{x_n}}}}}$$

Similarly, write

$$[x_0; x_1, x_2, \dots]$$

for the infinite continued fraction

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\cdots + \cfrac{1}{x_n + \cdots}}}}} .$$

Variations of this notation, and many other notations for continued fractions, are used in the literature.

The theory of continued fraction expansions of real numbers is based on the following "obvious" fact, which can be easily proved by induction:

Proposition.

Let $x_1, \ldots, x_n$ be positive real numbers, and x_0 a real number. Then the expression

$$x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{\cdots + \frac{1}{x_n}}}}}$$

unambiguously defines a real number by the usual rules of algebra. If $x_0, \ldots, x_n$ are rational, then the number defined by the expression is rational.

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Otherwise set $\beta_1 = x - a_0$ (fractional part of x), and $\alpha_1 = \frac{1}{\beta_1}$.
Then $\alpha_1 > 1$ and

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Continue inductively defining β_n, α_n, a_n for all n (or until the process terminates).

The finite or infinite sequence $[a_0; a_1, a_2, \ldots]$ is called the *continued fraction* (expression, expansion, decomposition) of x. The a_n are called the *terms* (or *elements*) of the continued fraction. The name "continued fraction" for this construction is appropriate, since the idea is to write

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We have, for all n (when defined):

$$x = [a_0; a_1, \dots, a_{n-1}, \alpha_n]$$

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$$

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$a_2 = 3$, and the process terminates, giving

$$\frac{10}{7} = 1 + \frac{1}{2 + \frac{1}{3}} = [1; 2, 3]$$

Example. Let φ be the "golden ratio"

$$\varphi = \frac{\sqrt{5}+1}{2} = 1.618\cdots$$

φ satisfies the relation

$$\frac{1}{\varphi} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2} = \varphi - 1$$

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So the continued fraction expansion of φ is

$$[1; 1, 1, 1, \ldots]$$

The rational numbers

$$r_n = \frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$$

are called the *continued fraction convergents* to x.

The numbers a_n, r_n, p_n, q_n, α_n, β_n depend on x; we will write them $a_n(x)$, etc., when it is necessary to specify the dependence on x.

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$$a_n(x) = a_k(\alpha_{n-k}(x)) \text{ for } 0 \le k \le n$$

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$r_0 = 1,\ r_1 = [1; 2] = 1 + \frac{1}{2} = \frac{3}{2} = 1.5;\ p_1 = 3,\ q_1 = 2.$
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$p_n = F_{n+2}$ and $q_n = F_{n+1}$, where F_n is the n'th Fibonacci number

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

The Gauss Map

The primary connection between continued fractions and dynamics is via the *Gauss Map*

$$f : (0,1] \to (0,1] \qquad f(x) = \frac{1}{x} \mod 1 = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

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The Gauss map is the "generic" chaotic map: any map with "fractal" behavior can be modeled using iterates of it.

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so $f(x)$ has continued fraction

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In fact, for $x = \frac{p}{q}$ the continued fraction construction is really the same as the Euclidean Algorithm for determining the greatest common divisor of p and q.

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It is known (Lamé's Theorem) that the number of steps to termination is bounded by

$$5 \log_{10}(m)$$

which is roughly 5 times the number of decimal digits in m.

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So $a = a_0(\frac{p}{q})$, $\alpha_1(\frac{p}{q}) = \frac{q}{b}$, etc.

Recursive Formulas

Let $[a_0; a_1, a_2, \ldots]$ be an infinite continued fraction expression, with $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. For each n set

$$r_n = \frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n] \ .$$

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Proposition.

For any $n \geq 2$,

(i) $p_n = p_{n-2} + a_n p_{n-1}$ and $q_n = q_{n-2} + a_n q_{n-1}$.

(ii) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$.

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In particular, p_n and q_n are relatively prime, and $q_n > q_{n-1}$. So

$$r_n - r_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}} = \frac{(-1)^{n+1}}{q_n q_{n-1}} \ .$$

In particular, these formulas hold for a sequence arising as the continued fraction of a real number.

Corollary.

(i) Let $[a_0; a_1, \ldots]$ be a sequence with $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$, and let (r_n) be the corresponding sequence of finite continued fractions. Then

$$r_0 < r_2 < r_4 < \cdots < r_5 < r_3 < r_1$$

and $|r_n - r_{n-1}| = \frac{1}{q_n q_{n-1}}$ for $n \geq 1$.
(ii) Let $x \in \mathbb{R}$ with nonterminating continued fraction expansion $[a_0; a_1, \ldots]$, and (r_n) the corresponding sequence of finite continued fractions. Then

$$r_0 < r_2 < r_4 < \cdots < x < \cdots < r_5 < r_3 < r_1$$

and $r_n \to x$. In fact, for $n \geq 1$,

$$|x - r_n| < |r_{n+1} - r_n| = \frac{1}{q_n q_{n+1}} < \frac{1}{a_{n+1} q_n^2} \leq \frac{1}{q_n^2}$$

(ii) is why r_n is called the n'th *continued fraction convergent* to x, and we may justifiably write

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}} .$$

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Corollary.

If x and y are distinct real numbers, their continued fraction expansions are different.

Conversely, under the conditions of (i), by the Nested Intervals Property there is a unique $x \in \mathbb{R}$ satisfying

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Then the continued fraction expansion of x is $[a_0; a_1, \ldots]$.

Proposition.

Let $x \in \mathbb{R}$ have continued fraction expansion of length at least $n+1$, and (r_k) the corresponding convergents. If y is between r_n and r_{n+1}, then $a_k(y) = a_k(x)$ for $0 \leq k \leq n$.

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Theorem.

The continued fraction expansion establishes a one-one correspondence between the set $\mathbb{I}$ of irrational numbers and the set of sequences $[a_0; a_1, a_2, \ldots]$ with $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for $n \geq 1$. If (x_m) is a sequence of irrational numbers and x an irrational number, then $x_m \to x$ if and only if, for each n, there is an m_0 such that $a_n(x_m) = a_n(x)$ for all $m \geq m_0$.

The last theorem can be rephrased in topology terms:

Corollary.

The map $x \mapsto [a_0(x); a_1(x), a_2(x), \dots]$ is a homeomorphism of $\mathbb{I}$ onto $\mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ with the product topology.

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If f is the Gauss map, and

$$x = [0; a_1, a_2, \ldots]$$

then

$$f(x) = [0; a_2, a_3, \ldots]$$

The last theorem can be rephrased in topology terms:

Corollary.

The map $x \mapsto [a_0(x); a_1(x), a_2(x), \ldots]$ is a homeomorphism of $\mathbb{I}$ onto $\mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ with the product topology.

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So when the Gauss map is transferred to $\mathbb{N}^{\mathbb{N}}$, it becomes the shift map

$$(a_1, a_2, \ldots) \mapsto (a_2, a_3, \ldots)$$

Periodic Continued Fractions

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$$\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, \ldots]$$

$$\sqrt{61} = [7; 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, \ldots]$$

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$$\sqrt{61} = [7; 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, \ldots]$$

The fractional parts of these continued fractions are periodic. The last term in the period is twice the integer part, and the preceding segment of the period is palindromic.

Homework Assignment: Compare the continued fraction expansion of $\sqrt{n}$ with the Newton's rule approximation to the positive root of

$$f(x) = x^2 - n$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = \frac{1}{2}\left(x_k + \frac{n}{x_k}\right)$$

$$x_0 = \lfloor \sqrt{n} \rfloor$$

and explain the relationship.

These numbers, along with the golden ratio, have repeating continued fraction expansions. They are also *quadratic numbers*, irrational numbers which are roots of quadratic polynomials with integer coefficients.

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Let x be an irrational number. The continued fraction expansion of x is eventually periodic if and only if x is a quadratic number.

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The direction $\Rightarrow$ was proved by Euler in 1737, and the more difficult direction $\Leftarrow$ by Lagrange in 1770.

There is no algebraic number of degree > 2 whose continued fraction expansion is known. It is not even known for any such algebraic number whether the terms of the continued fraction expansion are bounded or unbounded!

Other Continued Fraction Examples

Euler also proved that

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots, 1, 2k, 1, \ldots] \ .$$

This is not periodic, but has a simple pattern.

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In contrast, the continued fraction expansion of π is

$$[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2,$$

$$1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1,$$

$$4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1,$$

$$7, 1, 2, 3, 7, 1, 2, 1, 1, 12, 1, 1, 1, 3, 1, 1,$$

$$8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4,$$

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$$16, 1, 161, 45, 1, 22, 1, 2, 2, 1, 4, 1, 2, \ldots]$$

and no pattern is known (although there are other regular continued-fraction type representations of π).

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The largest term among the first 180 million is

$$a_{11504930} = 878783625 \ .$$

Efficient Approximation by Rational Numbers

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An "efficient" approximation is an approximation which is close by some measure depending on the size of the denominator of the rational number when expressed in lowest terms.

For example, we could ask:

Question.

For an irrational number x, and a natural number n, what is the rational number with denominator $\leq n$ which most closely approximates x?

The theory of continued fractions gives a complete answer to this question.

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Definition.

Let x be an irrational number. A rational number $\frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ is a *good approximation* to x if

$$\left| x - \frac{p'}{q'} \right| > \left| x - \frac{p}{q} \right|$$

for all $p', q' \in \mathbb{Z}$, $1 \leq q' \leq q$, $(p', q') \neq (p, q)$.
A rational number $\frac{p}{q}$ with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ is a *best approximation* to x if

$$\left| q'x - p' \right| > |qx - p|$$

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A best approximation to x is obviously a good approximation to x.

Theorem.

The continued fraction convergents $\frac{p_n}{q_n}$ to an irrational number x for $n \geq 1$ are precisely the best approximations to x, i.e. every continued fraction convergent is a best approximation, and every best approximation is a continued fraction convergent.

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Not every good approximation is a continued fraction convergent, but all good approximations to x are *secondary convergents*, numbers of the form $[a_0; a_1, \ldots, a_{n-1}, c_n]$, where $[a_0; a_1, \ldots]$ is the continued fraction expansion of x and $1 \leq c_n \leq a_n$. If $a_n > 1$ and $a_n/2 < c_n \leq a_n$, then the secondary convergent is a good approximation; if a_n is even and $c_n = a_n/2$, the secondary convergent is sometimes, but not always, a good approximation.

The next theorem is an important result about continued fractions for number theory purposes, and further justifies the statement that the continued fraction convergents to a real number are the "best" rational approximants.

Theorem.

Let x be an irrational number, and, for $n \in \mathbb{N}$, let $r_n = \frac{p_n}{q_n}$ be the n'th convergent to x.

(i) If $n \geq 2$, then at least one of the following inequalities holds:

$$|x - r_{n-1}| < \frac{1}{2q_{n-1}^2} \text{ or } |x - r_n| < \frac{1}{2q_n^2} .$$

(ii) Conversely, if $r = \frac{p}{q}$ $(p, q \in \mathbb{Z})$ satisfies

$$|x - r| < \frac{1}{2q^2}$$

then $r = r_n$ for some n.

Actually, one can do a little better:

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Let x be an irrational number, and, for $n \in \mathbb{N}$, let $r_n = \frac{p_n}{q_n}$ be the n'th convergent to x. Then for infinitely many n (at least one of every three consecutive n),

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The example of the golden ratio (Exercise ()) shows that $\sqrt{5}$ is the largest possible constant M for which $|x - r_n| < \frac{1}{M q_n^2}$ for infinitely many n in general.

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For example, since $a_4(\pi) = 292$, we have that $r_3(\pi) = \frac{355}{113}$ satisfies

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{292 \cdot 113^2} \approx 2.7 \times 10^{-7}$$

so $\frac{355}{113}$ agrees with π to (at least, in fact exactly) 6 decimal places. This approximation was known to Chinese astronomer Zu Chongzhi in the fifth century.

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Since $a_2(\pi) = 15$, even the well-known approximation $r_1(\pi) = [3; 7] = \frac{22}{7}$ is accurate within $\frac{1}{15 \cdot 7^2} = \frac{1}{735} = .00136 \cdots$.

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A theorem of Lochs says that for almost all $x \in \mathbb{I}$ (in the sense of Lebesgue measure), $r_n(x)$ asymptotically agrees with x to about n decimal places.

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$$\pi^4 = [97; 2, 2, 3, 1, 16539, 1, 6, 7, 6, 8, 6, 3, 9, 1, 1, 1, 18, 1, \dots]$$

so we have a very good approximation

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whose continued fraction expansion is

$$[0; 8, 9, 1, 149083, 1, 1, 1, 4, 1, 1, 1, 3, 4, 1, 1, 1, 15, a_{18}, 6, 1, 1, 21, 1, \dots]$$

where a_{18} is a 166 digit number.

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Despite the large numbers near the beginning, a (numerical) statistical analysis of this continued fraction suggests that it eventually behaves "typically."

Probabilistic Distribution of Continued Fraction Terms

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To make this precise, let

$$E_n^{(k)} = \{x \in (0,1) : a_n(x) = k\}$$

and let $P_n^{(k)}$ be the Lebesgue measure of $E_n^{(k)}$.
Then $P_n^{(k)}$ is a reasonable interpretation of the "probability" $P(a_n = k)$ that $a_n(x) = k$ for a randomly chosen x.

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The case $n = 1$ is the easiest to analyze. We have

$$E_1^{(k)} = \left(\frac{1}{k+1}, \frac{1}{k}\right]$$

and so

$$P_1^{(k)} = \frac{1}{k(k+1)} = \frac{1}{k^2+k} \ .$$

Thus with high probability $a_1(x)$ is small for a randomly chosen x: in fact, the probability that $a_1(x) \geq k$ is the Lebesgue measure of $(0, 1/k] = 1/k$.

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Theorem.

For any n and k,

$$\frac{1}{3k^2} < P_n^{(k)} < \frac{2}{k^2} \ .$$

Note that the bounds are independent of n.
So $P_n^{(k)} = O(\frac{1}{k^2})$ for all n and k.

Asymptotic Behavior of Continued Fractions

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Let $\phi : \mathbb{N} \to [1, \infty)$ be a function. Then
(i) If $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ diverges, then the set of $x \in \mathbb{I}$ for which $a_n(x) = O(\phi(n))$ as $n \to \infty$ has Lebesgue measure 0. In particular, the set of $x \in \mathbb{I}$ for which $(a_n(x))$ is bounded has Lebesgue measure 0.

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(ii) If $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ converges, then $a_n(x) = o(\phi(n))$ as $n \to \infty$ for almost all $x \in \mathbb{I}$, i.e. the set of x for which this fails has Lebesgue measure 0.

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(ii) If $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ converges, then $a_n(x) = o(\phi(n))$ as $n \to \infty$ for almost all $x \in \mathbb{I}$, i.e. the set of x for which this fails has Lebesgue measure 0.

Thus, almost all irrational numbers have many a_n which are "large" but which do not grow too rapidly.

In contrast, Khinchin proved the following remarkable theorem, which is essentially a version of the Strong Law of Large Numbers:

Theorem.

For almost all $x \in \mathbb{I}$, the geometric mean of the a_n approaches a constant

$$\kappa = \prod_{k=1}^{\infty} \left[1 + \frac{1}{k(k+2)}\right]^{\log_2 k} = \prod_{k=1}^{\infty} k^{\log_2\left[1+\frac{1}{k(k+2)}\right]} = 2.685452\cdots$$

i.e. $\lim_{n\to\infty} [a_1(x)a_2(x)\cdots a_n(x)]^{1/n} = \kappa$.

So, for almost all irrational numbers x, "most" of the $a_n(x)$ are small.

However, the *arithmetic* mean of the a_n's almost always diverges: since $\sum \frac{1}{n \log n}$ diverges, for almost all x we have $a_n(x) > n \log n$ for infinitely many n, and thus, for almost all x,

$$\frac{1}{n}\sum_{k=1}^{n} a_k(x) > \log n$$

for infinitely many n.

Khinchin's result is more general:

Theorem.

Let $f : \mathbb{N} \to [0, \infty)$ be a function for which there exist positive constants C and δ such that

$$f(k) < Ck^{1/2-\delta}$$

for all k. Then, for almost all $x \in \mathbb{I}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(a_k(x)) = \sum_{k=1}^{\infty} f(k) \log_2 \left[1 + \frac{1}{k(k+2)}\right] .$$

The previous theorem is the case $f(k) = \log k$.

Another interesting case is to let $f(k_0) = 1$ for some k_0 and $f(k) = 0$ for $k \neq k_0$. This case shows that the number k_0 appears with the same asymptotic relative frequency in the continued fraction expansion of almost all numbers, and the frequency is

$$\log_2(k_0^2 + 2k_0 + 1) - \log_2(k_0^2 + 2k_0) \ .$$

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For example, for almost all x, the relative frequency of 1 in the continued fraction expansion of x is

$$\log_2 4 - \log_2 3 = .4150 \cdots$$

i.e. asymptotically about 41.5% of the terms in the continued fraction expansion of x are 1's. Similarly, about 17% are 2's, 9.3% are 3's, etc.

Can you explicitly describe a subset of $\mathbb{R}$ which is not a Borel set?

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Let D be the set of real numbers x such that there exists a strictly increasing sequence (r_n) in $\mathbb{N}$ for which $a_{r_n}(x)$ divides $a_{r_{n+1}}(x)$ for all n.

D is a large subset of $\mathbb{R}$ (it contains all x which have infinitely many 1's in its continued fraction expansion, and hence $\mathbb{R} \setminus D$ has Lebesgue measure 0).

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D is not a Borel set in $\mathbb{R}$.

In fact, D is an analytic subset of $\mathbb{R}$. $\mathbb{R} \setminus D$ is not analytic. These are perhaps the simplest such subsets of $\mathbb{R}$ to explicitly describe.

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Let x be an algebraic number of degree $n > 1$. Then there is a constant C, $0 < C < 1/2$, such that

$$|x - r| > \frac{C}{q^n}$$

for all rational numbers $r = \frac{p}{q}$.

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for all rational numbers $r = \frac{p}{q}$.

This result is due to Liouville in 1844. The proof uses the Mean Value Theorem of calculus.

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Definition.

An irrational number x is a *Liouville number* if, for every n, there are $p, q \in \mathbb{Z}$ with $q \geq 2$ and

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^n} .$$

By the theorem, every Liouville number is transcendental. Liouville numbers are precisely the numbers x for which infinitely many $a_n(x)$ are very large compared to n.

For example, let

$$x = \sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000 \cdots \ .$$

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For each n, we can take $\frac{p}{q} = \sum_{k=1}^{n} 10^{-k!}$. This was Liouville's original example, and the first explicit number to be proved transcendental.
At that time (before Cantor proved that $\mathbb{R}$ is uncountable), it was not known that transcendental numbers existed.

The set L of Liouville numbers is quite an interesting subset of $\mathbb{R}$. It is very "small" in a measure-theoretic sense, but "large" in a topological (category) sense:

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In fact, there appears to be no foolproof way to distinguish between algebraic and transcendental numbers by a statistical analysis of the pattern of the terms in the continued fraction expansion.

Can you explicitly describe a subset of $\mathbb{R}$ which is not a Borel set?

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Let D be the set of real numbers x such that there exists a strictly increasing sequence (r_n) in $\mathbb{N}$ for which $a_{r_n}(x)$ divides $a_{r_{n+1}}(x)$ for all n.

D is a large subset of $\mathbb{R}$ (it contains all x which have infinitely many 1's in its continued fraction expansion, and hence $\mathbb{R} \setminus D$ has Lebesgue measure 0).

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In fact, D is an analytic subset of $\mathbb{R}$. $\mathbb{R} \setminus D$ is not analytic. These are perhaps the simplest such subsets of $\mathbb{R}$ to explicitly describe.