Classification of Simple C*-Algebras

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The Classification Theorem

Theorem:

The Elliott invariant is a complete isomorphism invariant for separable infinite-dimensional simple unital UCT C*-algebras of finite nuclear dimension: if A and B are separable infinite-dimensional simple unital UCT C*-algebras of finite nuclear dimension, and $Ell(A) \cong Ell(B)$, then $A \cong B$.

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Prehistory: J. Glimm, O. Bratteli, J. Cuntz, W. Krieger, M. Pimsner, D. Voiculescu, ...

Factors: J. von Neumann, F. Murray, R. Powers, D. McDuff, A. Connes, ...

C*-Algebras

Definition:

A C*-algebra is a complex Banach algebra A, with an involution * satisfying

$$(x+y)^* = x^* + y^*\,,\ (\lambda x)^* = \bar{\lambda} x^*\,,\ (xy)^* = y^* x^*\,,\ (x^*)^* = x$$

for all $x, y \in A, \lambda \in \mathbb{C}$, and satisfying the C*-axiom

$$\|x^* x\| = \|x\|^2 \text{ for all } x \in A\ .$$

Ingredients:
(i) Algebraic structure: complex vector space, ring, adjoint
(ii) Topological structure: norm

Standard Examples

1. $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, or any norm-closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ (*concrete C*-algebra*). In particular, $\mathbb{M}_n = M_n(\mathbb{C}) \cong \mathcal{B}(\mathbb{C}^n)$.

2. $C(X)$, X a compact Hausdorff space.
C*-Algebras are "noncommutative topological spaces."

C*-Algebras are not necessarily unital. In this talk we will restrict attention to unital C*-algebras, and all homomorphisms (*-homomorphisms) will be assumed to be unital too. Such homomorphisms are automatically norm-decreasing, and isomorphisms are automatically isometric.

Applications

Traditional:

Group representations
Mathematical physics (quantum mechanics)
Dynamical systems

Modern:

Topology and geometry
Free probability
ETC.

A simple (unital) C*-algebra is a unital C*-algebra with no nontrivial ideals.
Model example: $\mathbb{M}_n$.

$$a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

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A *trace* on a unital C*-algebra A is a linear functional τ on A such that $0 \leq \tau(x^*x) = \tau(xx^*)$ for all $x \in A$ and $\tau(1) = 1$. Then $\tau(xy) = \tau(yx)$ for all $x, y \in A$. Write $T(A)$ for the set of traces on A.

$$\tau(a) = \frac{1}{n} \sum_{k=1}^{n} a_{kk}$$

Projections and Equivalence

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$$p = p^* = p^2 \ .$$

Projections in $\mathcal{B}(\mathcal{H})$ are just orthogonal projections onto subspaces.

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Projections p and q in a C*-algebra A are *equivalent* (in A) if there is an $x \in A$ with $x^*x = p$, $xx^* = q$.
Two projections in $\mathcal{B}(\mathcal{H})$ are equivalent if and only if their range subspaces have the same dimension.
If τ is a trace on A and p and q are projections in A, and $p \sim q$, then $\tau(p) = \tau(q)$.
The unique trace τ on $\mathbb{M}_n$ takes the values

$$\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$$

on projections. These values parametrize the equivalence classes.

Von Neumann Algebras and Factors

A *von Neumann algebra* is a weakly closed unital *-subalgebra of $\mathcal{B}(\mathcal{H})$.
A *factor* is a von Neumann algebra with trivial center.

Murray and von Neumann showed that there are three types of factors which are simple C*-algebras:

1. Finite-dimensional matrix algebras $\mathbb{M}_n$ (*Type* I_n)
2. Infinite-dimensional factors with a trace (*Type* II_1)
3. Factors in which all nonzero projections are equivalent (*Type III*)

There are also two other kinds (*Type* I_∞, *Type* II_∞) which are not simple C*-algebras.

Murray and von Neumann showed that "pathological" factors of these types all exist.

They also proved that on a separable Hilbert space, there is a unique (up to isomorphism) hyperfinite (approximately finite dimensional) II_1 factor.

There are many non-hyperfinite II_1 factors (McDuff), which are still far from being classified.

There are many Type III factors. The approximately finite dimensional ones are classified (Powers, Connes, Krieger, Haagerup).

As C*-algebras, infinite-dimensional factors are not nice (e.g. nonseparable, nonnuclear). But the classification of approximately finite-dimensional factors can be taken as a model for the classification of some analogous class of simple C*-algebras. The model is quite imperfect and can be misleading; the C* situation is *much* more complicated.

As C*-algebras, infinite-dimensional factors are not nice (e.g. nonseparable, nonnuclear). But the classification of approximately finite-dimensional factors can be taken as a model for the classification of some analogous class of simple C*-algebras. The model is quite imperfect and can be misleading; the C* situation is *much* more complicated.

The structure of the projections is one of the biggest issues. Von Neumann algebras have *lots* of projections, separable C*-algebras not so many (only countably many equivalence classes).

For example, in a II_1 factor, there is a unique trace τ, which takes all values in $[0, 1]$ on projections (and parametrizes the equivalence classes).
"A II_1 factor is a *continuous factor* and gives a *continuous geometry*."

UHF Algebras

The first analogous class of simple C*-algebras were studied by Glimm around 1960.

If we have a sequence of C*-algebras and (unital) embeddings

$$A_1 \to A_2 \to \cdots$$

the embeddings are automatically isometries, so the "union" ("inductive limit") has a natural *-algebra structure and norm. [Technicality: this "union" must be completed to make a C*-algebra.]

Example: let $A_n = \mathbb{M}_{2^n}$, with embeddings

$$a \mapsto diag(a,a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

The inductive limit is called the *UHF algebra of type* 2^∞, or *CAR algebra*, denoted $\mathbb{M}_{2^\infty}$. It is an infinite-dimensional simple C*-algebra.

$\mathbb{M}_{2^\infty}$ has a unique trace τ, and the values τ takes on projections are exactly the set of dyadic rationals in $[0,1]$. These values parametrize the equivalence classes.

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If instead we take $A_n = \mathbb{M}_{3^n}$ with embeddings $a \mapsto diag(a, a, a)$, we get a UHF algebra $\mathbb{M}_{3^\infty}$. $\mathbb{M}_{3^\infty}$ also has a unique trace τ, and the values τ takes on projections are exactly the set of triadic rationals in $[0, 1]$, again parametrizing the equivalence classes. Thus $\mathbb{M}_{3^\infty} \not\cong \mathbb{M}_{2^\infty}$.

By varying the sizes of the matrix algebras at each step, and hence the multiplicities of the embeddings, uncountably many isomorphism classes of UHF algebras can be constructed, one for each (noncyclic) additive subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. If G is such a group, corresponding C*-algebra has a unique trace taking precisely the values $G \cap [0,1]$ on projections (and parametrizing the equivalence classes).

The UHF algebras can also be classified by *supernatural numbers*

$$2^{r_2}3^{r_3}5^{r_5}\cdots$$

where infinitely many nonzero exponents and infinite exponents are allowed.

This was Glimm's classification of UHF algebras. In contrast to the hyperfinite II_1 case, there is no uniqueness and "number theory" comes into the classification.

AF Algebras

The next step is to allow each A_n to be a (finite) direct sum of matrix algebras. One obtains a much larger class of C*-algebras called *AF algebras*, not all simple. The analysis of the isomorphism classes is more complicated, but essentially combinatorial and manageable. O. Bratteli carried this out and described the isomorphism classes via *Bratteli diagrams*. There is a simple criterion for determining when a Bratteli diagram gives a simple AF algebra.

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Example: Let $A_n = \mathbb{M}_{4^{n-1}} \oplus \mathbb{M}_{4^{n-1}}$, with connecting maps

$$(a, b) \mapsto (diag(a, a, a, b), diag(a, b, b, b))$$

The inductive limit is a simple C*-algebra A with unique trace τ, which on projections takes exactly the dyadic rational values in $[0, 1]$. But τ does not parametrize the equivalence classes: $p = (1, 0)$ and $q = (0, 1)$ in $A_1 = \mathbb{C} \oplus \mathbb{C}$ both have trace $\frac{1}{2}$ but are not equivalent in A. Thus $A \ncong \mathbb{M}_{2^\infty}$ and is not a UHF algebra.

By modifying this construction, taking the multiplicity matrix at the n'th stage to be

$$\begin{bmatrix} m_n & 1 \\ 1 & m_n \end{bmatrix}$$

with $m_n \to \infty$ rapidly enough, a simple AF algebra is obtained which has more than one trace (two extremal traces).

If A is a (unital) C*-algebra, write $T(A)$ for the set of traces on A. $T(A)$ is a compact convex subset of the dual space of A with the topology of pointwise convergence on A, and is a *Choquet simplex*, which is metrizable if A is separable.

By allowing many summands at each stage, any metrizable Choquet simplex can be realized as the trace space of a simple AF algebra.

Another interesting example: Let θ be an irrational number in $[0,1]$, with continued fraction expansion

$$(0; m_1, m_2, \dots) \ .$$

Consider the simple AF algebra with two summands at each stage, $A_1 = \mathbb{C} \oplus \mathbb{C}$, with the n'th connecting map given by the matrix

$$\begin{bmatrix} m_n & 1 \\ 1 & 0 \end{bmatrix}$$

The inductive limit M_θ is a simple AF algebra with unique trace taking precisely the values $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0,1]$ on projections (and parametrizing the equivalence classes).

K_0-Theory

In 1976, G. Elliott recast Bratteli's classification in K-theoretic terms. This was the first true step in the classification program (the next step didn't come for about 15 years!)

Let A be a C*-algebra. Consider projections in matrix algebras over A. If p is a projection in $M_n(A)$, for $m > n$ identify p with the projection $diag(p, 0)$ in $M_m(A)$. Consider the equivalence relation on projections in matrix algebras over A generated by this identification and ordinary equivalence. There is a natural sum on equivalence classes: $[p] + [q] = [diag(p, q)]$ (this is well defined!) making the set $V(A)$ of equivalence classes into an abelian semigroup with identity [0].

Examples: $V(\mathbb{C}) \cong V(\mathbb{M}_n) \cong \mathbb{N} \cup \{0\}$.
$V(\mathbb{M}_{2^\infty})$ is the additive semigroup of nonnegative dyadic rationals.
$V(M_\theta)$ is $(\mathbb{Z} + \mathbb{Z}\theta) \cap [0, \infty)$.

The Grothendieck Group

If S is any abelian semigroup, an abelian group $G(S)$ can be defined as equivalence classes of formal differences of elements of S, just as $\mathbb{Z}$ can be constructed from $\mathbb{N}$.

Definition:

Let A be a (unital) C*-algebra. $K_0(A) = G(V(A))$.

To keep the semigroup $V(A)$ in the picture, put a preordering on $K_0(A)$ by taking the image $K_0^+(A)$ of $V(A)$ as the positive cone. In good cases this is a partial ordering.

Examples: $K_0(\mathbb{C}) \cong K_0(\mathbb{M}_n) \cong \mathbb{Z}$ with the usual ordering. $K_0(\mathbb{M}_{2^\infty})$ is the dyadic rationals with the usual ordering from $\mathbb{R}$. $K_0(M_\theta) \cong \mathbb{Z} + \mathbb{Z}\theta$ with its ordering as a subgroup of $\mathbb{R}$. If $\theta' \neq \theta$, then $K_0(M_\theta)$ and $K_0(M_{\theta'})$ are isomorphic as groups (both $\mathbb{Z}^2$) but usually not as ordered groups.

One more ingredient: $[1_A]$ is a positive element of the (pre)ordered group $(K_0(A), K_0^+(A))$, and is an *order unit*. The triple $(K_0(A), K_0^+(A), [1_A])$ is called the *scaled ordered K_0 group of A*.

Examples: $(K_0(\mathbb{M}_n), K_0^+(\mathbb{M}_n), [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, n)$.
$(K_0(\mathbb{M}_{2^\infty}), K_0^+(\mathbb{M}_{2^\infty}), [1]) \cong (\mathbb{D}, \mathbb{D}^+, 1)$.
$(K_0(\mathbb{M}_\theta), K_0^+(\mathbb{M}_\theta), [1]) \cong (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)^+, 1)$.

If A is inductive limit of $\mathbb{M}_{4^{n-1}} \oplus \mathbb{M}_{4^{n-1}}$ discussed earlier, then

$$(K_0(A), K_0^+(A), [1_A]) \cong (\mathbb{D} \oplus \mathbb{D}, \{(0,0)\} \cup (\mathbb{D}^{++} \oplus \mathbb{D}), (1,0))$$

If A is an AF algebra, the (scaled) ordered K_0 group of A is called the *(scaled) dimension group* of A.

Theorem (Elliott):

The scaled dimension group is a complete isomorphism invariant for an AF algebra: if A and B are AF algebras and $(K_0(A), K_0^+(A), [1_A]) \cong (K_0(B), K_0^+(B), [1_B])$, then $A \cong B$.

Dimension groups can be abstractly characterized: a scaled ordered group (G, G^+, u) is a dimension group if and only if it is countable, unperforated, and has the Riesz Interpolation Property (Effros-Handelman-Shen Theorem).

The AF algebra of a dimension group is simple if and only if the group is a simple ordered group: every nonzero positive element is an order unit.

AH algebras

The next step (around 1990) was to bring some topology into the picture. We consider inductive limits where the "building blocks" are not finite-dimensional (direct sums of matrix algebras), but finite direct sums of matrix algebras over commutative C*-algebras, i.e.

$$M_{m_k}(C(X_k))$$

where the X_k are compact metrizable spaces. Such a C*-algebra is an *AH algebra* ("approximately homogeneous"). There is a slightly larger class of "approximately subhomogeneous" (ASH) C*-algebras.

There is a well-understood and easily checked criterion for when an AH or ASH algebra is simple.

One might think the class of (simple) AH or ASH algebras is vastly larger than the class of AF algebras. There is in fact a big difference, but not so large that the AF classification results cannot be adapted to them, with one big restriction:

A simple AH algebra has *slow dimension growth* if it has an inductive system in which the sizes of the matrix algebras grow faster than the topological dimensions of the base spaces. There is rather compelling evidence from topological vector bundle theory that AH algebras with slow dimension growth should be nicely behaved, and that ones without slow dimension growth might not be. In fact, it is now known that for simple AH algebras, "slow dimension growth" is equivalent to "finite nuclear dimension."

In particular, AH algebras with base spaces of bounded dimension ("no dimension growth") have slow dimension growth.

An *AI algebra* is an AH algebra where all the base spaces X_k are $[0,1]$. The essential difference between AI algebras and AF algebras is that AF algebras have "enough" projections (*real rank 0*) and AI algebras need not have real rank zero.

In the non-real-rank-zero case, an additional invariant is needed. Each equivalence class $[p]$ in $V(A)$ defines a continuous affine function $\hat{p}$ on $T(A)$:

$$\hat{p}(\tau) = \tau(p)$$

which gives a well-defined order-preserving group homomorphism

$$\rho : K_0(A) \to Aff(T(A))$$

with $\rho([1_A])$ the constant function 1, where $Aff(K)$ is the set of continuous affine real-valued functions on a convex set K.

If A has real rank zero, $T(A)$ and ρ can be recovered from the ordered group $K_0(A)$, but not in the simple AI case.

We thus add $T(A)$ and ρ to the invariant.

Theorem (Elliott 1992):

The invariant $(K_0(A), K_0^+(A), [1_A], T(A), \rho)$ is a complete isomorphism invariant for simple AI algebras.

The range of the invariant was also characterized.

K_1-Theory

There is one more ingredient in general.

Definition:

Let A be a unital C*-algebra. Then

$$K_1(A) = \lim_{\to} U(M_n(A))/U(M_n(A))_0 = \lim_{\to} GL_n(A)/GL_n(A)_0 \ .$$

For the C*-algebras we are considering, $K_1(A) = U(A)/U(A)_0$.

Examples: $K_1(\mathbb{M}_n) = 0$ (U_n and $GL_n(\mathbb{C})$ are connected.) Thus if A is AF, $K_1(A) = 0$.
Any von Neumann algebra has trivial K_1.
$K_1(C([0,1]) = 0$. Thus if A is AI, $K_1(A) = 0$.
$K_1(C(\mathbb{T})) \cong \mathbb{Z}$.

K_1 does not enter the picture for AF or AI algebras. But the next class considered are A$\mathbb{T}$ algebras, where the building blocks are matrix algebras over $C(\mathbb{T})$, and the K_1 is needed in the invariant.

The invariant is now complete:

Definition:

Let A be a simple unital C*-algebra. The *Elliott invariant* of A is

$$Ell(A) = (K_0(A), K_0^+(A), [1_A], K_1(A), T(A), \rho) \ .$$

K_1 does not enter the picture for AF or AI algebras. But the next class considered are $A\mathbb{T}$ algebras, where the building blocks are matrix algebras over $C(\mathbb{T})$, and the K_1 is needed in the invariant.

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Let A be a simple unital C*-algebra. The *Elliott invariant* of A is

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Theorem:

The Elliott invariant is a complete isomorphism invariant for simple $A\mathbb{T}$ algebras.

This theorem was gradually proved in a series of long and difficult papers by several authors in the 1990s.

Irrational Rotation Algebras and Crossed Products

If A is a C*-algebra, G is a group, and α is a homomorphism from G into the group of automorphisms of A, there is a larger C*-algebra containing A and a group of unitaries isomorphic to G implementing the automorphisms, called the *crossed product*, denoted $A \times_\alpha G$.

Important special case: if G acts on a compact Hausdorff space X (i.e. (X, G) is a *dynamical system*), then there is an induced action α of G on $C(X)$, and $C(X) \times_\alpha G$ is a *transformation group C*-algebra*.

Transformation group C*-algebras are very important and come up naturally in many applications. There is an extensive theory of crossed products. In particular, there is an extensive (but incomplete) criterion for when a crossed product is simple.

Let $G = \mathbb{Z}$. Fix an irrational number θ in $[0, 1]$, and let α_θ be the action of $\mathbb{Z}$ on $\mathbb{T}$ by rotation by $2\pi\theta$. The crossed product $C(\mathbb{T}) \times_{\alpha_\theta} \mathbb{Z}$ is called the *irrational rotation algebra* with angle θ, denoted A_θ.

The irrational rotation algebras come up naturally in several contexts (dynamical systems, noncommutative geometry, quantum Hall effect).

Properties:

1. A_θ is simple.
2. $(K_0(A_\theta), K_0^+(A_\theta), [1]) \cong (\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)^+, 1)$ (Rieffel)
3. $K_1(A_\theta) = \mathbb{Z}^2$.
4. A_θ has real rank 0 (B.-Kumjian-Rørdam)
5. A_θ is an A$\mathbb{T}$ algebra (Elliott-Evans)

Interlude: Tensor Products and Nuclear C*-Algebras

If A and B are C*-algebras, there is a *tensor product* $A \otimes B$.

Examples: 1. $A \otimes \mathbb{M}_n \cong M_n(A)$. In particular, $\mathbb{M}_m \otimes \mathbb{M}_n \cong \mathbb{M}_{mn}$.
2. $C(X) \otimes B \cong C(X, B)$. $C(X) \otimes C(Y) \cong C(X \times Y)$.

If A and B are simple, then $A \otimes B$ is simple.

The theory of tensor products of C*-algebras has some pathologies in general. But there is a large class of C*-algebras for which tensor products are well behaved, called *nuclear C*-algebras*.

There are several other equivalent characterizations of nuclear C*-algebras, such as:

They are “approximately finite-dimensional” in an order-theoretic sense.

They are “amenable.”

Thus they form a very natural class of C*-algebras to consider.

Furthermore, all commutative or finite-dimensional C*-algebras are nuclear, and the class of nuclear C*-algebras is closed under

extensions

inductive limits

tensor products

crossed products by amenable or connected groups

In particular, all ASH (AH, AF) algebras are nuclear. Irrational rotation algebras are nuclear.

Purely Infinite C*-Algebras

A simple (unital) C*-algebra with at least one trace is *stably finite*. At the other extreme are the *purely infinite* C*-algebras, the C*-analog of Type III factors.

The most important examples were constructed by Cuntz in the 1970s. If $n > 1$, let O_n be the C*-algebra generated by n isometries with mutually orthogonal range projections adding to the identity. So O_n is generated by $\{s_1, \ldots, s_n\}$, where

$$s_k^* s_k = 1 \ \forall k, \quad \sum_{k=1}^{n} s_k s_k^* = 1$$

All C*-algebras generated by such elements are isomorphic.

Properties of O_n:

1. O_n is separable, simple, and purely infinite.
2. O_n has real rank zero.
3. O_n is nuclear.
4. $K_0(O_n) = \mathbb{Z}_{n-1}$ and $K_1(O_n) = 0$.

There is also an O_∞, generated by a sequence of isometries with mutually orthogonal range projections. O_∞ is also separable, simple, purely infinite, real rank 0, and nuclear; $K_0(O_\infty) \cong \mathbb{Z}$ and $K_1(O_\infty) = 0$.

The O_n $(2 \leq n \leq \infty)$ are called the *Cuntz algebras*.

Cuntz and Krieger defined a larger class of algebras, the *Cuntz-Krieger algebras* O_A, with similar properties.

If A is purely infinite, then $K_0^+(A)$ is the entire group $K_0(A)$, and thus the ordering is completely degenerate. And $T(A) = \emptyset$, so the Elliott invariant reduces to $(K_0(A), [1_A], K_1(A))$.

Using tensor products and inductive limits, it is not hard to construct a separable purely infinite simple nuclear C*-algebra whose Elliott invariant is isomorphic to (G_0, u, G_1), where G_0 and G_1 are arbitrary countable abelian groups and u is an arbitrary element of G_0.

Although these examples were known, before about 1990 none of the classification techniques seemed to touch the purely infinite case and few believed a classification was on the horizon or even possible. But then suddenly almost out of nowhere came:

Theorem (Kirchberg, Phillips):

The Elliott invariant is a complete isomorphism invariant for separable simple purely infinite nuclear C*-algebras satisfying the UCT.

This was a much more far-reaching result than any of the previous classification theorems since it did not require that the algebra be expressible as an inductive limit of any type of special building blocks. These C*-algebras are called *Kirchberg algebras*.

Trouble in Paradise

By the mid-1990s, the heady successes of the classification program so far swung expectations to the other extreme from the hopelessness of 15 years earlier: it was widely conjectured that the Elliott invariant might be a complete isomorphism invariant for all separable simple nuclear C*-algebras.

Then some troubling examples started to appear. First were some examples from J. Villadsen:

A simple AH algebra with stable rank greater than one.

A simple AH algebra with perforation in K_0.

These AH algebras did not, and could not, have slow dimension growth. They shook the conjecture but did not directly contradict it, since their Elliott invariants were not the same as any of the algebras so far classified.

Then A. Toms constructed a simple AH algebra with the same Elliott invariant as an AI algebra, but was not an AI algebra (it did not have *strict cancellation*). This was a clean counterexample to the conjecture.

M. Rørdam constructed an even worse counterexample: a separable simple nuclear C*-algebra with both finite and infinite projections. It was neither stably finite nor purely infinite, but had the same Elliott invariant as a Kirchberg algebra.

Hope thus faded for a classification theorem with clean hypotheses.

The Jiang-Su Algebra

Another technical counterexample to the broad conjecture, an infinite-dimensional simple ASH algebra with the same Elliott invariant as $\mathbb{C}$, turned out to be a blessing in disguise.

The *Jiang-Su algebra* Z is a simple C*-algebra which is an inductive limit of *dimension-drop* C*-algebras of the form

$$I_{pq} = \{f : [0,1] \to \mathbb{M}_p \otimes \mathbb{M}_q : f(0) \in \mathbb{M}_p \otimes 1, f(1) \in 1 \otimes \mathbb{M}_q\}$$

If p and q are relatively prime, I_{pq} has no nontrivial projections.

Z turns out to have a number of remarkable properties. For one, it is *strongly self-absorbing* (O_2, O_∞, and certain UHF algebras were the only previously known such algebras), and in particular $Z \cong Z \otimes Z$.

The notion of Z-stability was found to be fundamental: a C*-algebra A is *Z-stable* if $A \cong A \otimes Z$.

In the meantime, the classification theorems were extended to simple ASH algebras over base spaces of dimension ≤ 3, covering enough examples to exhaust all possibilities for the Elliott invariant (with weakly unperforated K_0).

There was, however one glaring weakness in the classification theorems in the stably finite case: they only applied to ASH algebras. Thus to show a simple C*-algebra was in the classifiable class it had to be shown that it could be written as a suitable inductive limit of nice building blocks.

This was extremely problematic, for example, in the case of crossed products. Some difficult work showed that a few types of transformation group C*-algebras were ASH, but broad results were not obtained.

The second phase of the classification program thus began around 2000 (some sporadic results were earlier), to eliminate any type of ASH hypothesis in the stably finite case.

One notable advance was H. Lin's definition and study of *Tracially AF* (TAF) algebras. This class includes many simple nuclear C*-algebras not obviously ASH, let alone AF.

Another important development was pursuing consequences of $\mathcal{Z}$-stability, which for simple AH algebras is equivalent to slow dimension growth, but also applies and is relevant even for C*-algebras which are not known to be ASH.

Quasidiagonality became an important part of the story, although it is still somewhat mysterious. Separable simple quasidiagonal C*-algebras are NF algebras (B.-Kirchberg).

The most important development was W. Winter's various notions of topological dimension for C*-algebras. There have been many incarnations, culminating in the notion of *nuclear dimension* (with J. Zacharias).

Nuclear dimension generalizes topological dimension: the nuclear dimension of $C(X)$ is exactly the topological dimension (covering dimension) of X. It is infinite for nonnuclear C*-algebras.

Nuclear dimension for simple nuclear C*-algebras seems to always be 1, 2, or ∞ (and no examples are known to have nuclear dimension 2). UCT Kirchberg algebras have nuclear dimension ≤ 2. The classifiable simple ASH algebras have nuclear dimension 1. Finite nuclear dimension is preserved under standard operations, including crossed products by $\mathbb{Z}$ (with the Rokhlin property).

The following appear to be equivalent (*Toms-Winter Conjecture*):

1. Finite nuclear dimension
2. $\mathcal{Z}$-Stability
3. Strict comparison

They are equivalent if $T(A)$ is a Bauer simplex and $\partial T(A)$ is finite-dimensional (highly nontrivial).

For a simple AH algebra, they are all equivalent to slow dimension growth and approximate divisibility.

The following theorem almost finished off the Classification Theorem:

Theorem (Elliott, Gong, Lin, Niu):

The Elliott invariant is a complete isomorphism invariant for separable simple stably finite (unital) C*-algebras satisfying the UCT and of finite nuclear dimension, in which every trace is quasidiagonal.

The final step was the result of Tikuisis, White, and Winter in 2016, confirming a conjecture of B.-Kirchberg:

Theorem:

Every trace on a stably finite separable nuclear simple UCT C*-algebra is quasidiagonal.

Putting the pieces together, we get the final Classification Theorem:

Theorem:

The Elliott invariant is a complete isomorphism invariant for separable infinite-dimensional simple unital UCT C*-algebras of finite nuclear dimension: if A and B are separable infinite-dimensional simple unital UCT C*-algebras of finite nuclear dimension, and $Ell(A) \cong Ell(B)$, then $A \cong B$.

Most of the many ingredient theorems are difficult technical results which were only gradually found and proved.

The invariants which can arise have also been cleanly characterized and exhausted.

Loose Ends

A few loose ends remain to be cleaned up:

1. Finish proving the Toms-Winter Conjecture.

2. Is every separable simple unital nuclear C*-algebra with real rank zero in the classifiable class?

3. Extend the results to nonunital (projectionless) simple C*-algebras.

4. Extend to nonsimple C*-algebras.

5. The elephant in the closet: the UCT.

The UCT

The UCT is a question about bivariant K-theory, roughly whether ordinary K-theory determines bivariant K-theory. It is called the *Universal Coefficient Theorem* since it is an analog of the UCT of topology/homological algebra.

The UCT is known to hold for a bootstrap class of nuclear C*-algebras including all standard constructions. If there is a nuclear C*-algebra not satisfying the UCT, it cannot be constructed using any known methods. The good news is that essentially any nuclear C*-algebra which arises "naturally" can be readily shown to be in the UCT class.

Although no one seems to have any idea how to approach the general UCT question, there have been some recent results indicating that the UCT hypothesis can be eliminated (i.e. is automatic) in the Classification Theorem in the stably finite case with finite nuclear dimension.