THE ALGEBRAIZATION OF DYNAMICS:

Amenability, Nuclearity, Quasidiagonality, and Approximate Finite Dimensionality

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In honor of

John von Neumann (1903-1957)

and Marshall Stone (1903-1989)

Theorem. Let A be a stable simple C*-algebra. Then the following are equivalent:

(i) A is algebraically simple.

(ii) A contains an infinite projection.

(iii) There is no finite nonzero dimension function on the Pedersen ideal $\mathcal{P}(A)$.

(iv) There is no nonzero (necessarily faithful) densely defined quasitrace on A.

If A is exact, these are also equivalent to

(v) There is no nonzero (necessarily faithful) densely defined trace on A.

(vi) $\tilde{A}$ contains an infinite projection.

SOME CONSEQUENCES

Corollary. Let A and B be simple C*-algebras. If B contains an infinite projection, so does $A \otimes B \otimes \mathbb{K}$.

Definition. A C*-algebra A is *completely projectionless* if $A \otimes B$ is projectionless for all C*-algebras B.

Examples. (i) $C_0(X) \otimes B$, X connected and noncompact, B arbitrary

(ii) Any [sub]contractible C*-algebra

Rigidity Property:

Corollary. A simple C*-algebra cannot be written as an inductive limit of completely projectionless C*-algebras.

Definition. Let x be an element of a C*-algebra A. Then x is a *scaling element* if $xx^* \neq x^*x$ and $xx^* \ll x^*x$, i.e. $(x^*x)(xx^*) = xx^*$.

Simple functional calculus arguments show:

Proposition. Let A be a C*-algebra. Then

(i) If A contains a scaling element then $\mathcal{P}(A)$ contains a scaling element x and $a \geq 0$ such that $\|x\| = \|a\| = 1$ and $a \ll x^*x$, $a \perp xx^*$.

(ii) If A contains an "approximate scaling element," then A contains a scaling element.

Proposition. If A is a unital C*-algebra containing a scaling element, then A contains an infinite projection.

Proof: If x is a scaling element in A as above, then $u = x + (1 - x^*x)^{1/2}$ is a nonunitary isometry.

A C*-algebra A containing a scaling element contains a nonzero projection: if $x \in A$ is as above, and $u = x + (1 - x^*x)^{1/2} \in \tilde{A}$, then $1 - uu^*$ is a nonzero projection in A.

A C*-algebra containing a scaling element need not contain an infinite projection.

Original proof of proposition showed:

If A contains a scaling element x, then there is a sequence (p_k) of mutually orthogonal, mutually equivalent nonzero projections in A, with $p_k \ll x^*x \ \forall k$.

Proposition. Let A be a simple C*-algebra. If A contains a scaling element, then A contains an infinite projection.

Proof: Let x be a scaling element in $\mathcal{P}(A)$, and (p_k) a sequence of projections as above. Let $u_k^* u_k = p_1$ and $u_k u_k^* = p_k$ for each k. There are $y_1, \ldots, y_n \in A$ with

$$x^* x = \sum_{k=1}^{n} y_k^* p_1 y_k = \sum_{k=1}^{n} y_k^* u_k^* p_k u_k y_k$$

$$= (\sum_{k=1}^{n} y_k^* u_k^*)(\sum_{k=1}^{n} p_k)(\sum_{k=1}^{n} u_k y_k) = c^* p c$$

Set $q = p + p_{n+1} = \sum_{k=1}^{n+1} p_k$. Then

$$q = q(x^* x)q = qc^* pcq = u^* u \text{ for } u = pcq$$

and $uu^* \leq p$, so q is infinite.

Proof of Theorem

(ii) $\Rightarrow$ (vi) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (iii) is Handelman.

(v) $\Rightarrow$ (iv) in exact case is Haagerup.

Add an additional condition:

(vii) A contains a scaling element.

Main proof: show (vii) $\Rightarrow$ (ii) (above),

(i) $\Leftrightarrow$ (vii), (iii) $\Rightarrow$ (vii).

Proofs are similar to proof of (vii) $\Rightarrow$ (ii).

(vii) $\Rightarrow$ (i) is true in nonstable case too.

(vi) $\Rightarrow$ (v): if $\tilde{A}$ contains a nonunitary isometry, then it has one of the form $1+y$ with $y \in \mathcal{P}(A)$; $y^*y - yy^* \gneqq 0$. If τ is a trace on $\mathcal{P}(A)$, then

$$\tau(yy^*) = \tau(y^*y) = \tau(yy^*) + \tau(y^*y - yy^*)$$

Hyponormal Elements

Definition. An element x in a C*-algebra is *hyponormal* if $xx^* \leq x^*x$.

If A is stable, simple, and exact, the conditions of the theorem are equivalent to:

(viii) $\mathcal{P}(A)$ contains a nonnormal hyponormal element.

It is unclear whether this is equivalent to

(ix) A contains a nonnormal hyponormal element.

It is also unclear whether (viii) $\Leftrightarrow$ (i)-(iv) [and also whether (vi) $\Leftrightarrow$ (i)-(iv)] in the nonexact case.

Construction of Projectionless C*-Algebras

Let X be a closed orientable 3-manifold. $H^3(X) \cong \mathbb{Z}$.

Let $D(X,k)$ be the stable continuous trace C*-algebra with spectrum X and Dixmier-Douady invariant k.

$D(X,k)$ is projectionless if $k \neq 0$.

If σ is an orientation-preserving homeomorphism of X, then there is an automorphism α of $D(X,k)$ inducing σ, for any k.

Twisted Double Embedding (B.-Kumjian)

Embed $D(X,k)$ into $M_2(D(X,k)) \cong D(X,k)$ by $x \mapsto diag(x, \alpha(x))$.

Iterate the construction, and let $A(X,k,\alpha)$ be the inductive limit. Then $A(X,k,\alpha)$ is projectionless if $k \neq 0$.

Proposition. If σ is minimal, then $A(X,k,\alpha)$ is simple.

The isomorphism class of $A(X, k, \alpha)$ probably depends only on σ (and k).

It depends on more than the homotopy class of σ, which determines $K_*(A(X, k, \alpha))$: it also depends on the space of invariant measures, which determines the trace space.

Perhaps the Elliott invariant is a complete invariant. But:

The opposite algebra of $D(X, k)$ is isomorphic to the algebra $D(X, -k)$.

If X has no orientation-reversing homeomorphisms (e.g. certain lens spaces), and $k \neq 0$, then $D(X, k)$ is not anti-isomorphic to itself.

If such an X has a minimal homeomorphism σ, then $A(X, k, \alpha)$ is a candidate for a simple nuclear (stable, projectionless) C*-algebra not anti-isomorphic to itself.

The Elliott invariant does not distinguish between A and A^{op}.

Now let $X = \mathbb{T}^3$. Then $K_*(D(\mathbb{T}^3, k))$ can be calculated by realizing $D(\mathbb{T}^3, k)$ as a mapping torus of an automorphism of $C(\mathbb{T}^2) \otimes \mathbb{K}$: if $k \neq 0$, then

$$K_0(D(\mathbb{T}^3, k)) \cong \mathbb{Z}^3$$

$$K_1(D(\mathbb{T}^3, k)) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_k$$

If σ is a minimal homeomorphism (e.g. an irrational rotation), then $K_*(A(X, k, \alpha))$ can be computed.

Now tensor with O_∞. From the Theorem, we obtain:

Corollary. $A(\mathbb{T}^3, k, \alpha) \otimes O_\infty$ contains an infinite projection. Thus $D(\mathbb{T}^3, k) \otimes O_\infty$ contains infinite projections.

In fact, a multiple of every element of

$$K_0(D(\mathbb{T}^3, k) \otimes O_\infty) \cong K_0(D(\mathbb{T}^3, k))$$

is representable by an infinite projection in the algebra $D(\mathbb{T}^3, k) \otimes O_\infty$.

If p is an infinite projection in $B = D(\mathbb{T}^3, k) \otimes O_\infty$, then $A = pBp$ is a locally trivial bundle algebra over $\mathbb{T}^3$ with fiber O_∞ (or something Morita equivalent to it.)

The bundle is nontrivial by K-theory.

One gets similar bundles over any closed orientable 3-manifold with a minimal homeomorphism (e.g. S^3).

Conclusion: There is interesting topology (nontrivial higher homotopy) in $Aut(O_\infty)$.

Question: What happens if use O_2 instead of O_∞? Are the bundles trivial? The K-theory is 0.